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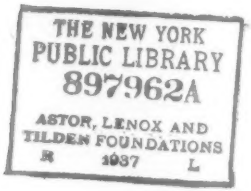

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CONTINUATION OF DIFFERENTIABLE FUNCTIONS THROUGH THE PLANE

By H. O. HIRSCHFELD (*Cambridge*)

[Received 18 July 1935]

Introduction

CONSIDER an open region in the (x, y)-plane and a function $f(x, y)$ having continuous derivatives up to the mth order inside the region. Suppose that the function $f(x, y)$ itself and all these derivatives have continuous limits on the boundary. The function will then be called 'C^m in the open region plus boundary'.

The problem is to construct, if possible, a function $F(x, y)$, which equals $f(x, y)$ in the original region and which has continuous mth derivatives in some region, to which the original one is interior. The possibility of constructing such a function is ensured by an interesting and more general theorem due to H. Whitney, provided the region fulfils a generalized convexity condition.† However, the extension of $f(x, y)$ given there would involve an infinite square subdivision of the complementary plane, even for a continuation through a single analytic boundary arc of the region.

Therefore, before starting on the main subject of this paper, we state two simpler methods of continuation. These will give us an extension $F(x, y)$ connected with the original function $f(x, y)$ by a finite formula. Thus it will now be possible to decide whether a subsidiary property of $f(x, y)$ is still true for $F(x, y)$ in the larger region.‡ We confine ourselves to C^2-continuation, though the method could be generalized so as to cover C^m-continuation; the restriction to the plane, however, is essential. Furthermore, we are concerned only with three special cases of Whitney's theorem: namely the continuation through a twice-differentiable arc and

† H. Whitney (*a*) *Trans. American Math. Soc.* 36 (1934), 63–89, (*b*) *Ann. of Math.* (2) 35 (1934), 482–5. The generalized convexity condition, called property P by H. Whitney, is the following: any pair of points P_1, P_2 in the region can be connected by a rectifiable curve lying in the region such that the ratio of the length of this curve to the distance $P_1 P_2$ is uniformly bounded for all pairs.

‡ e.g. the moduli of $F(x, y)$ and its derivatives depend in a simple way on those of $f(x, y)$ and the (twice-differentiable) arc of the region. Questions on the total variation, etc., of the highest derivatives are easily settled.

(as a generalization) through an arc fulfilling a Lipschitz condition.

Finally, we shall continue into the interior of the unit circle. For a wide class of regions these cases will be sufficient to extend C^2-functions through their boundaries by applying our simple methods successively to their boundary arcs.

The main part of this paper, however, is concerned with showing that mere continuity of the boundary arc is not sufficient for C^2-continuation through it. We shall see this from an interesting example: we shall construct a region E bounded by a continuous Jordan curve (which has even a continuous polar-representation $R = e(\phi)$) and a function $f(x, y)$ continuous in the closed region E, analytic in any interior sub-region and having continuous derivatives of any order in the closed region E.† But even C^1-continuation of this function $f(x, y)$ will be impossible beyond any sub-arc of the boundary of E. In other words: if we consider any region E^*, containing the above E, such that the points of a certain arc $R = e(\phi)$ $(\phi_0 \leqslant \phi \leqslant \phi_1)$ are interior points of E^*, then in E^* there is no continuous function $F(x, y)$ which equals the above $f(x, y)$ in E and has continuous first derivatives in E^*.‡

1. Continuation beyond a twice-differentiable arc. ('Reflection at an arc')

Consider in the (x, y)-plane the region R: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant q(x)$, where $q(x)$ is strictly positive and has a continuous second derivative in $(0, 1)$. Let $f(x, y)$ have continuous second derivatives in R: the continuity of f and its first derivatives is a consequence of this. Then there exists a function $F(x, y)$ having continuous second derivatives in the region S: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2q(x)$ and equal to $f(x, y)$ in R: and thus F and its first derivatives will be continuous in S too.

Proof. Let $\phi(x)$ be any function which has a continuous first

† A function will be said to have continuous derivatives of mth order in a closed region, if all mth derivatives are continuous in the interior of the region, and if the function itself and all its derivatives up to the mth order tend to continuous limits on the boundary of the region.

‡ In view of this example it seems to be somewhat artificial to define 'a function C^m in an open region plus boundary' by the possibility of its C^m continuation through the boundary as was done by H. Whitney, loc. cit. (*b*), 482.

derivative for all x and equals $f_y\{x, q(x)\}$ for $0 \leqslant x \leqslant 1$. Then

$$F(x,y) = \begin{cases} f(x,y) & (\text{in } R) \\ f\{x, 2q(x)-y\} - \int_0^{x+q(x)-y} \phi(\rho)\, d\rho + \int_0^{x-q(x)+y} \phi(\rho)\, d\rho & \\ & (\text{in } S-R), \end{cases}$$

is a function having the desired properties.

2. Continuation beyond an arc fulfilling a Lipschitz condition

If in § 1 we assume only that $q(x)$ is strictly positive† and fulfils a Lipschitz condition, i.e. if we assume that there is a constant M such that

$$|q(x_1)-q(x_2)| < M|x_1-x_2|$$

for any x_1, x_2 in the closed interval $(0, 1)$, then the assertion of § 1 is true also, even if S is any region containing R such that all boundary points of R are still boundary points of S with the exception of the points $\{x, q(x)\}$ for $0 < x < 1$.

Proof. The idea of the proof is to transform $f(x, y)$ into another twice-differentiable function which together with all its derivatives up to the second order vanishes on our boundary arc, then to define the transformed function as identically zero beyond the arc, and finally to reverse the transformations. These will consist in transforming the (x, y)-plane linearly and reversibly and in adding to the original function $f(x, y)$ summands twice-differentiable in the closed maps of S. Hence, whenever the transformed function has a desired continuation, so has the original function.

We start with the transformation

$$\eta = y+(2M+1)x, \qquad \xi = x,$$

by which S and R are transformed into S_1 and R_1 (say) respectively, M being the Lipschitz constant. Here R_1 is defined by

$$0 \leqslant \xi \leqslant 1, \qquad (2M+1)\xi \leqslant \eta \leqslant q(\xi)+(2M+1)\xi.$$

Obviously $\phi(\xi) = q(\xi)+(2M+1)\xi$ and $t(\xi) = \phi(\xi)-\xi$ are monotone increasing, and so are $\phi(\xi)-M\xi$ and $t(\xi)-M\xi$. Hence, if we denote the continuous monotone increasing inverse functions of $\phi(\xi)$ and $t(\xi)$ by

$$\xi = \psi(\eta), \qquad \xi = r(\eta),$$

† There is no difficulty in allowing $q(0) = 0$.

respectively, we obtain the Lipschitz conditions

$$|\psi(\eta_1)-\psi(\eta_2)| \leqslant |\eta_1-\eta_2|/M, \qquad |r(\eta_1)-r(\eta_2)| \leqslant |\eta_1-\eta_2|/M. \quad (2.1)$$

We now write $g(\xi, \eta) = f\{\xi, \eta-(2M+1)\xi\}$ and introduce

$$G(\xi, \eta) = g(\xi, \eta) - \int_0^{\eta} d\tau \int_0^{\tau} \Phi(\rho)\, d\rho,$$

where $\Phi(\rho)$ is any function, continuous for all ρ and equal to

$$g_{\xi\eta}\{\psi(\rho), \rho\}+g_{\eta\eta}\{\psi(\rho), \rho\} \text{ for } \phi(0) \leqslant \rho \leqslant \phi(1).$$

Thus $$G_{\xi\eta}\{\psi(\eta), \eta\}+G_{\eta\eta}\{\psi(\eta), \eta\} \equiv 0 \text{ for } \phi(0) \leqslant \eta \leqslant \phi(1). \quad (2.2)$$

The next transformation will be $X = \xi$, $Y = \eta-\xi$, by which S_1 and R_1 are transformed into S_2 and R_2 respectively. Here R_2 (say) is defined by

$$0 \leqslant X \leqslant 1, \qquad 2MX \leqslant Y \leqslant 2MX+q(X) \equiv t(X).$$

We now introduce $h(X, Y) = G(X, X+Y)$ and obtain by (2.2)

$$h_{XY}\{r(Y), Y\} \equiv 0 \text{ for } t(0) \leqslant Y \leqslant t(1). \quad (2.3)$$

Now, since, for any pair Y_1, Y_2 with $t(0) \leqslant Y_2 < Y_1 \leqslant t(1)$, the point $r(Y_1)$, Y_2 will lie in R_2 if Y_1-Y_2 is sufficiently small, we obtain for such pairs

$$\begin{aligned}\frac{h_Y\{r(Y_1), Y_1\}-h_Y\{r(Y_2), Y_2\}}{Y_1-Y_2} &= \frac{h_Y\{r(Y_1), Y_1\}-h_Y\{r(Y_1), Y_2\}}{Y_1-Y_2}+\frac{h_Y\{r(Y_1), Y_2\}-h_Y\{r(Y_2), Y_2\}}{Y_1-Y_2} \\ &= h_{YY}\{r(Y_1), Y^*\}+h_{YX}(X^*, Y_2)\frac{r(Y_1)-r(Y_2)}{Y_1-Y_2},\end{aligned}$$

where $$Y_2 < Y^* < Y_1, \qquad r(Y_2) < X^* < r(Y_1).$$

By (2.1) and (2.3) the last term tends to 0, if $Y_1 \to Y_2$ or $Y_2 \to Y_1$.

Hence $$\frac{d}{dY}[h_Y\{r(Y), Y\}] = h_{YY}\{r(Y), Y\} \text{ for } t(0) \leqslant Y \leqslant t(1). \quad (2.4)$$

Thus we are able to construct a function $\Psi(Y)$ equal to $h_Y\{r(Y), Y\}$ for $t(0) \leqslant Y \leqslant t(1)$ and having a continuous first derivative for all Y. We then introduce

$$H(X, Y) = h(X, Y) - \int_0^{Y} \Psi(\rho)\, d\rho.$$

Now this function $H(X, Y)$ will have continuous second derivatives, and the identities

$$H_Y\{r(Y), Y\} \equiv 0, \quad H_{YY}\{r(Y), Y\} \equiv 0, \quad H_{XY}\{r(Y), Y\} \equiv 0 \quad (2.5)$$

will hold for $t(0) \leqslant Y \leqslant t(1)$. The first is an immediate consequence of the definition of H and Ψ, the second follows with the help of (2.4), the third with the help of (2.3). But now the function $H\{X, t(X)\}$ has continuous first and second derivatives for $0 \leqslant X \leqslant 1$, namely

$$\frac{d}{dX}[H\{X, t(X)\}] = H_X\{X, t(X)\}; \frac{d^2}{dX^2}[H\{X, t(X)\}] = H_{XX}\{X, t(X)\}. \quad (2.6)$$

This we infer since the Lipschitz condition holds for $t(X)$ and from (2.5) by the same argument as above. By (2.6) we are able to construct a function $\Omega(X)$ having a continuous second derivative for all X and equal to $H\{X, t(X)\}$ for $0 \leqslant X \leqslant 1$. Thus finally by (2.5) and (2.6) the function

$$K(X, Y) = H(X, Y) - \Omega(X),$$

vanishes together with all its derivatives on $Y = t(X)$ $(0 \leqslant X \leqslant 1)$. Therefore $K(X, Y)$ will have continuous first and second derivatives in S_2, if defined as identically zero in $S_2 - R_2$. Reversing all the transformations, we obtain for $F(x, y)$ the expression

$$F(x, y) = \begin{cases} f(x, y) & (\text{in } R) \\ \Omega(x) + \int_0^{y+2Mx} \Psi(\rho)\, d\rho + \int_0^{y+(2M+1)x} d\tau \int_0^{\tau} \Phi(\rho)\, d\rho & (\text{in } S - R), \end{cases}$$

where $\Omega(x)$ is any function which has a continuous second derivative for all x and equals†

$$f\{x, q(x)\} - \int_0^{q(x)+2Mx} \Psi(\rho)\, d\rho - \int_0^{q(x)+(2M+1)x} d\tau \int_0^{\tau} \Phi(\rho)\, d\rho \text{ for } 0 \leqslant x \leqslant 1;$$

and $\Psi(\rho)$ is any function which has a continuous first derivative for all ρ and equals†

$$f_y\{r(\rho),\ \rho - 2Mr(\rho)\} - \int_0^{\rho+r(\rho)} \Phi(\lambda)\, d\lambda \text{ for } q(0) \leqslant \rho \leqslant q(1) + 2M;$$

and $\Phi(\rho)$ is any function which is continuous for all ρ and equals†

$$f_{xy}\{\psi(\rho),\ \rho - (2M+1)\psi(\rho)\} - 2Mf_{yy}\{\psi(\rho),\ \rho - (2M+1)\psi(\rho)\}$$
$$\text{for } q(0) \leqslant \rho \leqslant 2M + 1 + q(1).$$

† $q(x)$, $r(\rho)$, and $\psi(\rho)$ are representations of our boundary arc.

1′. Continuation into the interior of the unit circle

Let $f(\rho,\phi)$ have continuous second derivatives for $1 \leqslant \rho \leqslant 2$ and all ϕ, and let all these functions be periodic in ϕ. Then there exists a function $F(\rho,\phi)$ equal to $f(\rho,\phi)$ in the above ring and having continuous second derivatives for $0 \leqslant \rho \leqslant 2$ and all ϕ. If, in addition, we put $x = \rho\cos\phi$, $y = \rho\sin\phi$, then F will also have continuous second derivatives with regard to x, y in the circle with radius 2.

Proof. Obviously the function $f^*(\rho,\phi) = f(\rho,\phi)/\rho^3$ is also twice differentiable for $1 \leqslant \rho \leqslant 2$ and all ϕ. Hence the function

$$F^*(\rho,\phi) = \begin{cases} f^*(\rho,\phi) & (1 \leqslant \rho \leqslant 2;\ \text{all } \phi) \\ f^*(2-\rho,\phi) - \int\limits_0^{1-\rho+\phi} f_\rho^*(1,\tau)\,d\tau + \int\limits_0^{\rho-1+\phi} f_\rho^*(1,\tau)\,d\tau & \\ & (0 \leqslant \rho \leqslant 1;\ \text{all } \phi), \end{cases}$$

has continuous second derivatives in ρ,ϕ for $0 < \rho \leqslant 2$ and all ϕ; furthermore all these derivatives and F^* itself are bounded and periodic in ϕ. Thus, finally, the function $F(\rho,\phi) = \rho^3 F^*(\rho,\phi)$ has the properties wanted, if F and its derivatives are defined as zero at the origin.

If the whole boundary of a region locally admits a representation fulfilling a Lipschitz condition,† then, as may be stated without proof, C^2-functions may be extended through it by the above methods. We have to apply § 2 successively to suitably chosen arcs. We start with a subdivision of the original boundary and then take the contours of wedge-shaped gaps left between the first continuations, thus embedding the original region in a larger one. To fill up the whole plane, suitable twice-differentiable transformations may shorten the process considerably; and § 1′ has to be applied, if the original region is of higher connectivity.

3. Impossibility of continuation

We are now going to show that C^m-continuation through a continuous boundary arc is not always possible. Before, however, constructing an example as announced in the introduction, we state some simple regions showing that a cusp of the complementary plane may prevent a continuation into it.

† Every boundary point P is centre of a circle C_P such that all boundary points in C_P admit a representation $\eta_P = q_P(\xi_P)$ where the (ξ_P, η_P)-plane is connected with the (x, y)-plane by a reversible, twice-differentiable transformation and where $q_P(\xi_P)$ fulfils a Lipschitz condition.

(i) Consider, in the (x, y)-plane, the region $y \leqslant \sqrt[4]{|x|}$ and the function

$$f(x,y) = \begin{cases} y^3 & (0 \leqslant x;\ 0 \leqslant y \leqslant \sqrt[4]{|x|}), \\ 0 & (\text{elsewhere}). \end{cases}$$

Obviously $f(x, y)$ is twice differentiable in the closed region in question.† Now suppose there is a C^1-continuation in the neighbourhood of the origin, i.e. suppose there is a function $F(x, y)$ continuous together with at least $F_x(x, y)$ in a circle round the origin and equal to $f(x, y)$ for $y \leqslant \sqrt[4]{|x|}$. Then for sufficiently small positive x we should obtain by the mean-value theorem

$$[F(x, \sqrt[4]{|x|}) - F(-x, \sqrt[4]{|x|})]/2x = F_x(x^*, \sqrt[4]{|x|}) \qquad (-x < x^* < x),$$

but on the other hand we should have

$$[F(x, \sqrt[4]{|x|}) - F(-x, \sqrt[4]{|x|})]/2x = (\sqrt[4]{|x|})^3/2x = \tfrac{1}{2}x^{-\frac{1}{4}},$$

and this would contradict the continuity of F_x as $x \to 0$.

It is not an essential feature of our example that some derivative, higher than the second, should have a discontinuity in the original region. For the function

$$g(x,y) = \begin{cases} \exp(-y^{-2}) & (x \geqslant 0;\ y \geqslant 0), \\ 0 & (\text{elsewhere}), \end{cases}$$

has continuous derivatives of any order below the curve

$$|x|^{\frac{1}{2}} = \exp(-y^{-2}),$$

i.e. in the closed region R: $0 \leqslant |x| \leqslant 1$, $y \leqslant (-2/\log|x|)^{\frac{1}{2}}$ and, by the same argument as above, it cannot be continued C^1 through this boundary. It is not even essential for our example that the function should not be analytic everywhere inside R.‡ For instead of $g(x, y)$ we might use the function

$$h(r, \phi) = e^{-1/r^2} \cos \tfrac{1}{2}(\phi - \tfrac{1}{2}\pi),$$

(r, ϕ being polars about the origin) in R for the same argument.

But all these examples have *one* critical point only (a cusp at the origin), beyond which a continuation is impossible. Thus the question arises as to whether a condensation of these cusps might not give us the example mentioned in the introduction. This is actually the case. But peculiar geometrical difficulties arise: if a set of cusps has to be inserted into a boundary on an everywhere-dense set, how are

† i.e. is continuous and has continuous derivatives up to the second order in the closed region (see footnote †, p. 2). The behaviour at $y = \infty$ is of no relevance here.

‡ I am indebted to Mr. L. C. Young, who pointed out this to me and suggested a condensation employed below.

we to prevent the effect of any certain cusp from being 'extinguished' by the neighbouring ones? Let us first introduce the cusps in question. They are the 3-parametric families of continuous curves

$$R = q(\phi; \phi_0, R_0, \alpha)$$

$$\equiv \begin{cases} R_0 + \alpha\sqrt{8} \Big/ \sqrt{\left\{-\log\left(e^{-\frac{1}{2}} \frac{|\phi-\phi_0|}{2\pi}\right)\right\}} & (0 < |\phi-\phi_0| \leqslant \pi) \\ R_0 & (\phi = \phi_0) \end{cases} \tag{3.01}$$

for all positive R_0, and $\frac{1}{2} \leqslant \alpha \leqslant 1$. Here ϕ, R are polars about the origin and ϕ is measured in the positive sense of rotation from the x-axis.

Now we shall proceed in the following way: we shall not insert single cusps into the boundary but pairs of cusps, a sharp one and a blunt one having the same point (ϕ_0, R_0) as vertex and given by the curves $\qquad R = q(\phi; \phi_0, R_0, 1), \qquad R = q(\phi; \phi_0, R_0, \alpha_0),$

respectively. We shall then take care that all pairs of cusps inserted afterwards shall not reach the interior curve $R = q(\phi; \phi_0, R_0, \alpha_0)$ having the blunt cusp. Thus, though the original contour of the region, $R = q(\phi; \phi_0, R_0, 1)$ will be 'extinguished', the interior curve will always remain inside the region or at least on its boundary.

(ii) *Construction of the region.*

Let us first note some properties of the above families

$$R = q(\phi; \phi_0, R_0, \alpha).$$

$$\frac{d}{d\phi} q(\phi; \phi_0, R_0, \alpha) \to \begin{matrix} +\infty & (\phi \to \phi_0 + 0) \\ -\infty & (\phi \to \phi_0 - 0) \end{matrix} \Big\} \text{ independently of } R_0; \tag{3.02}$$

$$\left.\begin{array}{l} \left|\dfrac{dq}{d\phi}(\phi; \phi_0, R_0, \alpha)\right| \\ \text{is a monotone decreasing function of } |\phi-\phi_0| \text{ independent} \\ \text{of } \phi_0, R_0 \quad (0 < |\phi-\phi_0| \leqslant 2\pi; \alpha \text{ fixed}). \end{array}\right\} \tag{3.03}$$

Secondly, some notation has to be introduced. We put

$$\phi_{\nu n} = 2\pi\nu/2^n \qquad (\nu = 0, 1, 2, \ldots, 2^n; \text{ all } n);$$

we denote by $R_{\nu n}$ positive numbers such that $R_{\nu n} = R_{\mu m}$ whenever $\phi_{\nu n} = \phi_{\mu m}$, and we use in the course of the proof certain closed ϕ-intervals $\Delta_{\nu n}$† which always contain the point $\phi_{\nu n}$ as interior point.

† By 'ϕ-interval' we mean all ϕ-values between two given ϕ-values, the latter being included; by 'ϕ-angle' we mean all points (ϕ, R) in the (x, y)-plane, for which ϕ is in a certain ϕ-interval.

Now we start to construct by induction two sequences of functions $e_n(\phi)$ and $i_n(\phi)$ for $n = 1, 2, \ldots$, having the following properties:

$e_n(\phi)$ is continuous and periodic in $0 \leqslant \phi \leqslant 2\pi$ and this interval $(0, 2\pi)$ can be subdivided into a finite number of intervals such that on any of them $e_n(\phi)$ is equal to one of the 2^n-1 functions $q(\phi; \phi_{\nu n}, R_{\nu n}, 1)$ with certain $R_{\nu n}$ and $\nu = 1, 2, \ldots, 2^n-1$: in particular, there are in this subdivision 2^n-1 intervals of type $\Delta_{\nu n}$ such that

$$e_n(\phi) = q(\phi; \phi_{\nu n}, R_{\nu n}, 1)$$

on any $\Delta_{\nu n}$ for all $\nu = 1, 2, \ldots, 2^n-1$; (3.04)

$i_n(\phi)$ is defined in $(0, 2\pi)$ and equal to $q(\phi; \phi_{\nu n}, R_{\nu n}, \alpha_{\nu n})$ in $\Delta_{\nu n}$ for a suitable $\alpha_{\nu n}$ with $\frac{1}{2} \leqslant \alpha_{\nu n} < 1$ and the $R_{\nu n}$ of (3.04); (3.05)

$$1 \leqslant i_n(\phi) \leqslant i_{n+1}(\phi) \leqslant e_{n+1}(\phi) \leqslant e_n(\phi) \quad \text{in } (0, 2\pi); \tag{3.06}$$

in $i_n(\phi) \leqslant e_n(\phi)$ the sign of equality is only true for $\phi = \phi_{\nu n}$; $\nu = 1, 2, \ldots, 2^n-1$; (3.07)

$$e_n(\phi) - e_{n+1}(\phi) < 1/(n+1)^2; \tag{3.08}$$

$i_n(\phi)$ and $\dfrac{d}{d\phi} e_n(\phi)$ are continuous in $(0, 2\pi)$ except at the endpoints of the $\Delta_{\mu m}$ for $m = 1, 2, \ldots, n$ and $\mu = 1, 2, \ldots, 2^n-1$, where both of them may have ordinary discontinuities, and at $\phi = \phi_{\nu n}$, where only $\dfrac{d}{d\phi} e_n(\phi)$ is discontinuous, since, according to (3.04), $e_n(\phi)$ has a cusp there. (3.09)

We start with

$$e_1(\phi) = q(\phi; \pi, 1, 1), \qquad i_1(\phi) = q(\phi; \pi, 1, \tfrac{1}{2}), \qquad \Delta_{1,1} = (0, 2\pi),$$

and suppose that the first n pairs have been constructed. Let us then note two consequences first. In any closed set of values of ϕ not containing a point $\phi_{\nu n}$ we have, by (3.07) and (3.09),

$$0 < \delta < e_n(\phi) - i_n(\phi), \tag{3.10}$$

and, by (3.03), (3.04), (3.09),

$$|de_n(\phi)/d\phi| < M, \tag{3.11}$$

where δ and M depend, of course, on the set in question.

To construct the pair $e_{n+1}(\phi)$, $i_{n+1}(\phi)$, we consider an arbitrary but fixed ν with $0 \leqslant \nu \leqslant 2^n-1$ and introduce $\phi_i = \phi_{\nu n} + i\pi/2^{n+1}$ $(i = 1, 2, 3)$.

First of all we see that

$$\left.\begin{aligned} e_n(\phi) &\leqslant q\{\phi;\phi_2,e_n(\phi_2),1\} \qquad (\phi_1 \leqslant \phi \leqslant \phi_3),\\ &\text{the sign of equality holding only for } \phi=\phi_2. \end{aligned}\right\} \tag{3.12}$$

This is a consequence of (3.03), (3.04), (3.09) by the mean-value theorem applied to q and e_n on the intervals (finite in number) in which their derivatives are continuous. We now choose δ and M according to (3.10) and (3.11) for the interval (ϕ_1,ϕ_3) and consider the family $q(\phi;\phi_2,R,1)$ for some ϕ-interval $\bar{\phi}_1 \leqslant \phi \leqslant \bar{\phi}_2$ with $\phi_1 < \bar{\phi}_1 < \phi_2 < \bar{\phi}_2 < \phi_3$, which by (3.02) and (3.03) can be chosen such that on it

$$\left|\frac{d}{d\phi}q(\phi;\phi_2,R,1)\right| \geqslant M.$$

By (3.12) we know that

$$\epsilon = \min\Big\{[q\{\bar{\phi}_1;\phi_2,e_n(\phi_2),1\}-e_n(\bar{\phi}_1)],\\ [q\{\bar{\phi}_2;\phi_2,e_n(\phi_2),1\}-e_n(\bar{\phi}_2)],\quad \frac{2}{(n+1)^2},\quad 2\delta\Big\} \tag{3.13}$$

is positive, and hence by (3.01) we obviously have

$$q\{\bar{\phi}_1;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,1\}-e_n(\bar{\phi}_1) > 0,$$
$$q\{\bar{\phi}_2;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,1\}-e_n(\bar{\phi}_2) > 0.$$

On the other hand clearly

$$q\{\phi_2;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,1\}-e_n(\phi_2) = -\tfrac{1}{2}\epsilon < 0.$$

Hence remembering the continuity of both e_n and q we can find an interval $\psi_1 \leqslant \phi \leqslant \psi_2$ with $\bar{\phi}_1 \leqslant \psi_1 < \phi_2 < \psi_2 \leqslant \bar{\phi}_2$, such that

$$q\{\phi;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,1\}-e_n(\phi) \leqslant 0 \text{ in } (\psi_1,\psi_2), \tag{3.14}$$

and the sign of equality in (3.14) only holds for $\phi=\psi_1$ and $\phi=\psi_2$.† Furthermore we know that

$$\left|\frac{d}{d\phi}q\{\phi;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,1\}\right| > M \text{ in } (\psi_1,\psi_2). \tag{3.15}$$

But then we obtain by (3.11), (3.13), (3.14), (3.15)

$$-\min\{1/(n+1)^2,\delta\} < q\{\phi;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,1\}-e_n(\phi) \leqslant 0 \text{ in } (\psi_1,\psi_2). \tag{3.16}$$

Finally, by (3.10) and (3.16) we choose an α^* with $\frac{1}{2} \leqslant \alpha^* < 1$ such that

$$i_n(\phi) < q\{\phi;\phi_2,e_n(\phi_2)-\tfrac{1}{2}\epsilon,\alpha^*\} \text{ in } (\psi_1,\psi_2). \tag{3.17}$$

† ψ_1,ψ_2 are the roots of $q\{\phi;\phi_2,e(\phi_2)-\frac{1}{2}\epsilon,1\}-e_n(\phi)=0$ which lie next to ϕ_2.

We are now ready to call $\phi_2 = \phi_{\mu,n+1}$ (putting $\mu = 2\nu+1$)

$$e_n(\phi_2)-\tfrac{1}{2}\epsilon = R_{\mu,n+1}, \qquad (\psi_1,\psi_2) = \Delta_{\mu,n+1}, \qquad \alpha^* = \alpha_{\mu,n+1},$$

and to define, for $\phi_{\nu n} \leqslant \phi \leqslant \phi_{\nu+1,n}$,

$$\left.\begin{aligned} e_{n+1}(\phi) &= \begin{cases} q(\phi;\phi_{\mu,n+1},R_{\mu,n+1},1) \text{ in } \Delta_{\mu,n+1} \\ e_n(\phi)\ldots \text{ elsewhere in } (\phi_{\nu n},\phi_{\nu,n+1}) \end{cases} \\ i_{n+1}(\phi) &= \begin{cases} q(\phi;\phi_{\mu,n+1},R_{\mu,n+1},\alpha_{\mu,n+1}) \text{ in } \Delta_{\mu,n+1} \\ i_n(\phi)\ldots \text{ elsewhere in } (\phi_{\nu n},\phi_{\nu,n+1}) \end{cases} \end{aligned}\right\} (\nu = 0,\ldots,2^n-1).$$

Obviously, by (3.14), (3.16), (3.17) the properties (3.04),..., (3.09) are then satisfied for $n+1$.† For the sequence $e_n(\phi)$ we obtain by (3.06) and (3.08), for $m > n$,

$$0 \leqslant e_n(\phi)-e_m(\phi) \leqslant \sum_{\nu=n}^{m} 1/\nu^2.$$

Hence $\lim_{n\to\infty} e_n(\phi) = e(\phi)$, say, uniformly in ϕ, and thus $e(\phi)$ is continuous in $0 \leqslant \phi \leqslant 2\pi$ and by (3.04) satisfies

$$e(0) = e(2\pi), \qquad e(\phi_{\nu n}) = R_{\nu n} \quad (\nu = 1, 2,\ldots, 2^n-1; \text{ all } n). \tag{3.18}$$

Finally, we introduce the regions

$$E_n: \qquad 0 \leqslant \phi \leqslant 2\pi, \quad 0 \leqslant R \leqslant e_n(\phi),$$

for which the common part is seen to be the region

$$E: \qquad 0 \leqslant \phi \leqslant 2\pi, \quad 0 \leqslant R \leqslant e(\phi).$$

(iii) *Construction of the function $f(\phi, R)$ and proof of its property.*

It will now be convenient to introduce polars $r_{\nu n}$, $\theta_{\nu n}$, which have as centre the point $P_{\nu n} = (\phi_{\nu n}, R_{\nu n})$ constructed in (ii). Thus $r_{\nu n}(P)$ denotes the distance of an arbitrary point P from $P_{\nu n}$; $\theta_{\nu n}(P)$ denotes the angle by which the axis $\phi = \phi_{\nu n}$ has to be rotated about $P_{\nu n}$ in the positive sense of rotation in order that the point $\{\phi_{\nu n}, R_{\nu n}+r_{\nu n}(P)\}$ may become the point P. Any function of $\theta_{\nu n}, r_{\nu n}$ can be written as a function of the polars (ϕ, R) or of the cartesians (x, y) as long as $r_{\nu n} > 0$: its value for $r_{\nu n} = 0$ will be understood to be the limit, if any, for $r_{\nu n} \to 0$.

Thus any of the 2^n-1 functions

$$f_{\nu n}(\phi, R) = 2^{-2n}\exp\{-1/[r_{\nu n}(P)]^2\}\cos\{\tfrac{1}{2}\theta_{\nu n}(P)\}, \qquad P = (\phi, R) \tag{3.19}$$

has continuous derivatives with regard to x, y of any order in the

† We have to define $\Delta_{2\nu,n+1} = \Delta_{\nu,n}$ – common part of $(\Delta_{\nu n}, \Delta_{2\nu+1,n+1})$ – common part of $(\Delta_{\nu n}, \Delta_{2\nu-1,n+1})$.

closed region E_n and thus *a fortiori* in the closed region E.† Furthermore each $f_{\nu n}$ is analytic in any region which has no point in common with the semi-axis $\phi = \phi_{\nu n}$, $R \geqslant R_{\nu n}$. Hence the function

$$f(\phi, R) = \sum_{n=1}^{\infty} \sum_{\substack{\nu=1 \\ 2 \nmid \nu}}^{2^n-1} f_{\nu n}(\phi, R), \tag{3.20}$$

has continuous (x, y)-derivatives of any order in the closed region E, is continuous itself there, and is analytic in any sub-region of E. But it is impossible to continue $f(\phi, R)$ together with $\frac{\partial}{\partial \phi} f(\phi, R)$ continuously into any region S which has so much as an arc $e^*(\phi)$ of the curve $e(\phi)$ in common with E.‡ To show this we suppose the contrary to be true, i.e. we suppose there is a function $F(\phi, R)$ equal to $f(\phi, R)$ in E which is continuous in $E+S$ and has a continuous derivative $\frac{\partial}{\partial \phi} F(\phi, R)$ there.§ Now by (3.18) we could find on the arc $e^*(\phi)$ an interior point of the form $(\phi_{\mu m}, R_{\mu m})$, where μ and m are integers and μ is odd. This point gives rise to a circle C with centre at $(\phi_{\mu m}, R_{\mu m})$ lying entirely in $E+S$. Now define

Δ_0 a ϕ-interval containing $\phi_{\mu m}$ such that for all ϕ in Δ_0 the points $\{\phi, e(\phi)\}$ are in C;

Δ_1 the common part of Δ_0, $\Delta_{\mu,m}$ and the interval

$$\phi_{\mu m} - 2\pi/2^{m+1} \leqslant \phi \leqslant \phi_{\mu m} + 2\pi/2^{m+1};$$

Δ_2 the sum of the two interior quarters of Δ_1;

T the common part of C and the closed angle ϕ in Δ_2.

We furthermore write

$$f(\phi, R) = \underbrace{\sum_{n=1}^{\infty} \sum_{\substack{\nu=1 \\ 2 \nmid \nu}}^{2^n-1} f_{\nu n}(\phi, R)}_{\phi_{\nu n} \text{ not in } \Delta_1} + \underbrace{\sum_{n=1}^{\infty} \sum_{\substack{\nu=1 \\ 2 \nmid \nu}}^{2^n-1} f_{\nu n}(\phi, R)}_{\phi_{\nu n} \text{ in } \Delta_1} = f_1(\phi, R) + f_2(\phi, R).$$

Now $f_1(\phi, R)$ obviously has continuous derivatives of any order in $E+T$. Thus, subtracting it from $F(\phi, R)$, we should arrive at a function $F_2(\phi, R) = F(\phi, R) - f_1(\phi, R)$ equal to $f_2(\phi, R)$ in E and continuous together with $\frac{\partial}{\partial \phi} F_2(\phi, R)$ in $E+T$. Finally, we have by (3.05)

† See footnote †, p. 2.

‡ More exactly, S is supposed to be a region such that $E+S$ has the properties of E^* considered in the introduction.

§ The continuity of $F_R(\phi, R)$ will not be used to arrive at the contradiction.

and (3.06) for ϕ in Δ_1,

$$e(\phi) \geqslant i_m(\phi) = q(\phi; \phi_{\mu m}, R_{\mu m}, \alpha_{\mu m}),$$

which shows that the curve $R = i_m(\phi)$ lies in E for ϕ in Δ_1. Now the process by which the desired contradiction will be arrived at is the following. We choose on this curve $R = i_m(\phi)$ (ϕ in Δ_1) two points

$$\begin{aligned} P(\eta) &= \{\phi_{\mu m}+\eta, i_m(\phi_{\mu m}+\eta)\} = \{\phi^+(\eta), R^*(\eta)\} \\ Q(\eta) &= \{\phi_{\mu m}-\eta, i_m(\phi_{\mu m}-\eta)\} = \{\phi^-(\eta), R^*(\eta)\} \end{aligned} \qquad (\eta > 0)$$

and estimate the difference-ratio

$$[f_2(\phi^+, R^*)-f_2(\phi^-, R^*)]/(\phi^+-\phi^-).$$

It will be shown that this ratio tends to $+\infty$ as $\eta \to 0$. But on the other hand the ratio ought to be equal to

$$\frac{\partial F_2}{\partial \phi}\{\bar{\phi}(\eta), R^*(\eta)\},$$

taken for a suitable mean-value $\bar{\phi}(\eta)$. This will clearly contradict the continuity of $\dfrac{\partial F_2}{\partial \phi}$ as $\eta \to 0$. To estimate the difference

$$f_2(\phi^+, R^*)-f_2(\phi^-, R^*),$$

we compute any of its summands $f_{\nu n}(\phi^+, R^*)-f_{\nu n}(\phi^-, R^*)$ ($\phi_{\nu n}$ in Δ_1) separately. We first assume

$$\phi_{\nu n} > \phi^+ \quad \text{or} \quad \phi_{\nu n} < \phi^-.$$

Then the function $f_{\nu n}(\phi, R)$ is an analytic function of the corresponding polars $\theta_{\nu n}$, $r_{\nu n}$ in a suitable convex region containing the arc $R = R^*$, $\phi^- \leqslant \phi \leqslant \phi^+$. Hence by the mean-value theorem

$$\begin{aligned} &2^{2n}[f_{\nu n}(\phi^+, R^*)-f_{\nu n}(\phi^-, R^*)] \\ &\quad = \exp[-1/\{r_{\nu n}(P)\}^2]\cos\{\tfrac{1}{2}\theta_{\nu n}(P)\}-\exp[-1/\{r_{\nu n}(Q)\}^2]\cos\{\tfrac{1}{2}\theta_{\nu n}(Q)\} \\ &\quad = \exp\{-1/r_1^2\}(2/r_1^3)\cos\tfrac{1}{2}\theta_1[r_{\nu n}(P)-r_{\nu n}(Q)]- \\ &\qquad\qquad -\exp\{-1/r_1^2\}\tfrac{1}{2}\sin\tfrac{1}{2}\theta_1[\theta_{\nu n}(P)-\theta_{\nu n}(Q)], \end{aligned}$$

where $$r_{\nu n}(P) \lessgtr r_1 \lessgtr r_{\nu n}(Q), \qquad 0 \leqslant \theta_1 \leqslant 2\pi.$$

But, if we introduce $r_2 = \min\{r_{\nu n}(P), r_{\nu n}(Q)\}$, we have

$$\begin{aligned} \exp\{-1/r_1^2\} &= e^{-1/r_1^2}+e^{-1/r_1^2}-e^{-1/r_1^2} \\ &= e^{-1/r_1^2}+e^{-1/r_1^2}(2/r_3^3)(r_1-r_2), \end{aligned}$$

where $r_1 \geqslant r_3 \geqslant r_2$. Hence we obtain

$$\frac{2^{2n}|f_{\nu n}(\phi^+, R^*)-f_{\nu n}(\phi^-, R^*)|}{\phi^+-\phi^-} \leqslant \frac{|r_{\nu n}(P)-r_{\nu n}(Q)|}{\phi^+-\phi^-} e^{-1/r_1^2}(2/r_1^3)|\cos\tfrac{1}{2}\theta_1|+$$
$$+\frac{|r_{\nu n}(P)-r_{\nu n}(Q)|}{\phi^+-\phi^-} e^{-1/r_3^2}(1/r_3^3)|\sin\tfrac{1}{2}\theta_1|\,.\,|\theta_{\nu n}(P)-\theta_{\nu n}(Q)|+$$
$$+\frac{|\theta_{\nu n}(P)-\theta_{\nu n}(Q)|r_2}{\phi^+-\phi^-} e^{-1/r_3^2}(1/2r_2)|\sin\tfrac{1}{2}\theta_1|. \quad (3.21)$$

Now, since by elementary geometry

$$|r_{\nu n}(P)-r_{\nu n}(Q)| \leqslant (\phi^+-\phi^-)R^*(\eta),$$

$|\theta_{\nu n}(P)-\theta_{\nu n}(Q)|r_2 < (\frac{1}{2}\pi)(\phi^+-\phi^-)R^*(\eta)$,† $0 \leqslant \theta_{\nu n}(P),\ \theta_{\nu n}(Q) \leqslant 2\pi$, it is obvious that the right-hand side of (3.21) will be smaller than some constant multiple of $R^*(\eta)$. Therefore we obtain

$$\sum_{n=1}^{\infty}\sum_{\nu=1}^{2^n}{}^{\overbrace{\phi_{\nu n}\text{ in }\Delta_1\text{ and }\phi_{\nu n}>\phi^+ \atop \text{or }\phi_{\nu n}<\phi^-}} |f_{\nu n}(\phi^+, R^*)-f_{\nu n}(\phi^-, R^*)|/(\phi^+-\phi^-) \leqslant C', \quad (3.22)$$

where, of course, C' does not depend on η since $R^*(\eta)$ is bounded. We now consider among the $\phi_{\nu n}$ in Δ_1 those, for which

$$\phi^- \leqslant \phi_{\nu n} \leqslant \phi^+.$$

For these we infer from (3.19)

$$f_{\nu n}(\phi^+, R^*) \geqslant 0 \quad \text{and} \quad f_{\nu n}(\phi^-, R^*) \leqslant 0. \quad (3.23)$$

Hence combining (3.22) and (3.23) we obtain

$$\frac{f_2(\phi^+, R^*)-f_2(\phi^-, R^*)}{\phi^+-\phi^-} \geqslant \frac{f_{\mu m}(\phi^+, R^*)-f_{\mu m}(\phi^-, R^*)}{\phi^+-\phi^-} - C'. \quad (3.24)$$

Now, since $q(\phi; \phi_{\mu m}, R_{\mu m}, \alpha_{\mu m}) \geqslant R_{\mu m}$, we have

$$\theta_{\mu m}(P) = 2\pi-\theta_{\mu m}(Q) \leqslant \tfrac{1}{2}\pi,$$

and thus, putting $\rho(\eta) = r_{\mu m}\{P(\eta)\} = r_{\mu m}\{Q(\eta)\}$, we arrive at

$$\begin{aligned} f_{\mu m}\{\phi^+(\eta), R^*(\eta)\} &\geqslant (1/\sqrt{2})(2^{-2m})e^{-1/\{\rho(\eta)\}^2}, \\ f_{\mu m}\{\phi^-(\eta), R^*(\eta)\} &\leqslant -(1/\sqrt{2})(2^{-2m})e^{-1/\{\rho(\eta)\}^2}, \end{aligned} \quad (3.25)$$

† Firstly, $(\phi^+-\phi^-)R^*(\eta) > \overline{PQ} \geqslant |r_{\nu n}(P)-r_{\nu n}(Q)|$. Secondly, consider the triangle $P_{\nu n}$, P, Q with sides $\overline{PQ}$, $r_{\nu n}(Q)$, $r_{\nu n}(P)$ and assume without loss of generality $r_{\nu n}(P) \geqslant r_{\nu n}(Q) = r_2$. Furthermore, consider the triangle $P_{\nu n}$, Q, P^*, where P^* lies on $\overline{P_{\nu n}P}$ and has distance $r_{\nu n}(Q)$ from $P_{\nu n}$. Then we have $\overline{PQ} \geqslant \overline{P^*Q} \geqslant (2/\pi)\times$ length of the smaller arc of a circle through Q and P^*.

But solving the equations $R^*(\eta) = i_m\{\phi^+(\eta)\}$ and $R^*(\eta) = i_m\{\phi^-(\eta)\}$ for $\phi^+(\eta)-\phi_{\mu m}$ and for $\phi^-(\eta)-\phi_{\mu m}$ respectively, we obtain by (3.01)

$$\phi^+(\eta)-\phi^-(\eta) \leqslant 4\pi/e^{-\frac{1}{2}} \times \exp[-2/\{R_{\mu m}-R^*(\eta)\}^2].$$

Hence since clearly $R^*(\eta)-R_{\mu m} \leqslant \rho(\eta)$ we finally arrive at

$$\phi^+(\eta)-\phi^-(\eta) \leqslant 4\pi e^{\frac{1}{2}}[e^{-1/\{\rho(\eta)\}^2}]^2. \tag{3.26}$$

Combining now (3.24), (3.25), (3.26), we see that

$$\lim_{\eta \to 0} \frac{f_2\{\phi^+(\eta), R^*(\eta)\}-f_2\{\phi^-(\eta), R^*(\eta)\}}{\phi^+(\eta)-\phi^-(\eta)} = +\infty, \tag{3.27}$$

since, by the continuity of $i_m(\phi)$, certainly $\lim_{\eta \to 0} \rho(\eta) = 0$. On the other hand, under the above assumption of a C^1-continuation $F_2(\phi, R)$ in $E+T$, we should obtain by the mean-value theorem

$$\frac{f_2\{\phi^+(\eta), R^*(\eta)\}-f_2\{\phi^-(\eta), R^*(\eta)\}}{\phi^+(\eta)-\phi^-(\eta)} = \frac{\partial F_2}{\partial \phi}\{\bar{\phi}(\eta), R^*(\eta)\},$$

for some $\bar{\phi}(\eta)$ with $\phi^-(\eta) \leqslant \bar{\phi}(\eta) \leqslant \phi^+(\eta)$ and this would clearly contradict (3.27) or the continuity of $\dfrac{\partial F_2}{\partial \phi}$ in $E+T$.

Finally, I should like to express my thanks to Mr. L. C. Young for his kind help throughout the preparation of this paper.

ON THE INTEGERS WHICH ARE THE TOTIENT OF A PRODUCT OF THREE PRIMES

By PAUL ERDŐS (*Manchester*)

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THROUGHOUT this paper $p, q, r, p', q', r', s, P$ are used to denote prime numbers; ϵ an arbitrarily small positive number; n all sufficiently large integers, i.e. $n > n(\epsilon)$; $N(p, m)$ the number of the primes p not exceeding m and belonging to a defined set. The C denote positive absolute constants, not always the same in each occurrence. I prove the following

THEOREM. *If $f(n)$ is the number of solutions of the equation*

$$(p-1)(q-1)(r-1) = n \tag{1}$$

in primes p, q, r no two of which are equal, then

$$\overline{\lim_{n\to\infty}} f(n) = \infty.$$

I believe, but cannot prove, that a similar result holds for the solution of

$$(p-1)(q-1) = n.$$

We require the following

LEMMA.* *If the primes p are such that $p-1$ has more than $(1+\epsilon)\log\log n$ or less than $(1-\epsilon)\log\log n$ different prime factors, then*

$$N(p, n) = o\left(\frac{n}{\log n}\right). \tag{2}$$

This result is included in the more general one that, if P_k denotes a prime such that $P_k - 1$ has exactly k different prime factors, then

$$N(P_k, n)\log^2 n < Cn(C+\log\log n)^{k+3}.$$

From this is deduced exactly as in my paper quoted above that

$$\sum_{k<(1-\epsilon)\log\log n} N(P_k, n) + \sum_{k>(1+\epsilon)\log\log n} N(P_k, n) = o\left(\frac{n}{(\log n)^{1+\delta}}\right) \tag{3}$$

where $\delta = \delta(\epsilon) > 0$.

I prove from (3) that, if P denotes a prime such that $P-1$ has more than $(1+\epsilon)\log\log P$ or less than $(1-\epsilon)\log\log P$ factors, then

* P. Erdős, *Quart. J. of Math.* (Oxford), 6 (1935), 205–13.

$\sum P^{-1}$ converges. It suffices to show that

$$N(P,n) = o\left(\frac{n}{(\log n)^{1+\delta'}}\right) \tag{4}$$

where δ' is a positive constant. In (4) either $P \leqslant \sqrt{n}$, i.e. there are at most $\sqrt{n}$, values of P; or $n \geqslant P > \sqrt{n}$, and then

$$\log\log n \geqslant \log\log P > \log\log n - 1,$$

and so

$$(1+\epsilon)\log\log P > (1+\tfrac{1}{2}\epsilon)\log\log n,$$

$$(1-\epsilon)\log\log P < (1-\tfrac{1}{2}\epsilon)\log\log n.$$

Hence the P's exceeding $\sqrt{n}$ are included among the primes $Q\,(<n)$ for which $Q-1$ has more than $(1+\frac{1}{2}\epsilon)\log\log n$ or less than $(1-\frac{1}{2}\epsilon)\log\log n$ different prime factors. Then from (3), with $\delta' = \delta(\frac{1}{2}\epsilon)$,

$$N(P,n) \leqslant o\left(\frac{n}{(\log n)^{1+\delta'}}\right) + n^{\frac{1}{2}} = o\left(\frac{n}{(\log n)^{1+\delta'}}\right).$$

Typify by A the positive integers not exceeding n such that

$$pqr = A,$$

where no two of the primes p, q, r are equal, and $p-1$, $q-1$, $r-1$ each have more than $(1-\epsilon)\log\log n$ factors. I prove that

$$N(A,n) > C\frac{n}{\log n}. \tag{5}$$

Denote by p' (or q', r') the primes such that $p'-1$ has more than $(1-\epsilon)\log\log n$ different prime factors. Take the primes, say r', less than $n/p'q'$ for arbitrary and unequal p', q', and multiply them by $p'q'$. The integers $p'q'r'$ belong to the A's, and each A can be obtained at most six times in this way. Hence

$$6N(A,n) \geqslant \sum_{p',q'} N'\left(r', \frac{n}{p'q'}\right), \tag{6}$$

the summation being extended over all different p', q', and N' denoting the omission of p', q' among the r' in calculating N. It suffices for our object to take only those p', q' for which

$$n^{\frac{1}{3}} < p', q' < n^{\frac{1}{2}}.$$

I prove now that

$$\sum_{n^{\frac{1}{3}} < p' < n^{\frac{1}{2}}} \frac{1}{p'} > C. \tag{7}$$

For

$$\sum_{p \leqslant n} \frac{1}{p} = \log\log n + C + o(1),$$

and so
$$\sum_{n^{\frac{1}{4}}<p<n^{\frac{1}{2}}} \frac{1}{p} > C. \tag{8}$$

The primes p in this sum such that $p-1$ has less than $(1-\epsilon)\log\log n$ different prime factors occur among the primes $P\,(n^{\frac{1}{4}} < P < n^{\frac{1}{2}})$ which are such that $P-1$ has less than $(1-\frac{1}{2}\epsilon)\log\log P$ different prime factors. For clearly

$$(1-\epsilon)\log\log n < (1-\tfrac{1}{2}\epsilon)\log\log P,$$

since $$\log\log n - \log 8 < \log\log P < \log\log n - \log 4.$$

Hence, since the series $\sum P^{-1}$ converges,

$$\sum_{n^{\frac{1}{4}}<P<n^{\frac{1}{2}}} P^{-1} < \epsilon, \quad \text{say},$$

for arbitrarily small positive ϵ, and n greater than some $n(\epsilon)$. Then (7) follows on omitting the P from the p in (8). On squaring (7),

$$\sum_{n^{\frac{1}{4}}<p',q'<n^{\frac{1}{2}}}^{p'\neq q'} \frac{1}{p'q'} > C, \tag{9}$$

the omission of the terms in which $p' = q'$ being allowable, since $\sum 1/p^2$ converges. On subtracting from the number of primes s less than $n/p'q'$ the number of those for which $s-1$ has less than $(1-\frac{1}{2}\epsilon)\log\log\frac{n}{p'q'}$ different prime factors, i.e. $o\left(\frac{n}{p'q'}\Big/\log\frac{n}{p'q'}\right)$ from (2), by replacing ϵ by $\frac{1}{2}\epsilon$ and n by $n/p'q'$, we have

$$N\left(r', \frac{n}{p'q'}\right) > \frac{Cn}{p'q'\log n}.$$

Hence, from (6),

$$6N(A,n) \geqslant \sum_{n^{\frac{1}{4}}<p',q'<n^{\frac{1}{2}}}^{p'\neq q'} \frac{Cn}{p'q'\log n} \geqslant \frac{Cn}{\log n},$$

by (9).

Denote now by B_1, B_2,... the different integers in the set $\phi(A)$, where A does not exceed n and ϕ denotes Euler's ϕ-function. The B's are clearly of the form

$$(p'-1)(q'-1)(r'-1).$$

I prove that
$$N(B,n) = o(n/\log n). \tag{10}$$

Define the *quadratic part* of an integer $I = p^\alpha q^\beta \ldots$ as the product of the powers p^α with indices exceeding unity. Split the B's into two classes B_1, B_2 according as their quadratic part has respectively more than or not more than $\epsilon\log\log n$ different prime factors. Obviously

the integers B_1 have a divisor which is composed of prime factors, in number exactly $[\epsilon \log\log n]$ $(= j$, say), each occurring with a power exceeding unity. Hence

$$N(B_1, n) < n\left(\sum_{p,\alpha} \frac{1}{p^\alpha}\right)^j < \frac{nC^j}{j!} < n\left(\frac{C}{j}\right)^j = o\left(\frac{n}{\log n}\right),$$

where the summation extends to all primes p and all indices α exceeding unity so that the double series converges.

Each of the integers in B_2 is of the form

$$(p'-1)(q'-1)(r'-1),$$

where $p'-1$, $q'-1$, $r'-1$ each have at least $(1-\epsilon)\log\log n$ different prime factors, and so at least $(3-3\epsilon)\log\log n$ prime factors, not necessarily all different. But from the definition of B_2, $p'-1$, $q'-1$ can have as common factors at most $\epsilon \log\log n$ different primes; and similarly for $q'-1$, $r'-1$, etc. Hence each integer in B_2 has at least $(3-6\epsilon)\log\log n$ different prime factors and so at least $2^{(3-6\epsilon)\log\log n}$ divisors. But

$$\sum_{t=1}^{n} d(t) < Cn \log n,$$

and so* $$N(B_2, n) < Cn \log n / 2^{(3-6\epsilon)\log\log n} = o\left(\frac{n}{\log n}\right),$$

if we now suppose ϵ taken so small that

$$2^{3-6\epsilon} > e^2 = (2{\cdot}71\ldots)^2.$$

Hence $$N(B, n) = N(B_1, n) + N(B_2, n) = o(n/\log n).$$

This is (10).

Then, from (5), $N(A, n) \geqslant \frac{1}{\epsilon} N(B, n)$ for every positive constant ϵ, if $n > n(\epsilon)$. Hence at least one of the B's less than n is represented at least $1/\epsilon$ times in the form $\phi(N)$ for every $1/\epsilon$ if $n > n(\epsilon)$. This concludes the proof of the main theorem.

* It is here that the method breaks down for the solution of

$$(p-1)(q-1) = n.$$

A GENERALIZATION OF THE KALUZA-KLEIN FIELD THEORY

By B. HOFFMANN (*Princeton*)

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1. Introduction

In the general theory of relativity the time-like geodesics of the space-time continuum are the trajectories of test particles—including photons—in the gravitational field. If, however, an electromagnetic field is present in addition to the gravitational field, and if the test body carries electric charge in addition to mass, its trajectory will no longer be a geodesic of the space-time of the general theory of relativity. In the five-dimensional theory† of Kaluza and Klein the time-like geodesics of the cylindrical five-dimensional continuum can be shown to be the trajectories of charged mass-points in the combined gravitational and electromagnetic field. The Kaluza-Klein theory has been modified in two ways in order to avoid the introduction of a fifth dimension which can have no physical significance. Veblen and Hoffmann‡ showed that a satisfactory theory could be constructed in which the basic geometry was the generalized projective geometry of four dimensions, while Einstein and Mayer§ have used a four-dimensional space with every point of which is associated a five-dimensional auxiliary flat-space in the manner of a tangent space. The two theories have been shown to be very closely related,‖ and, in each, certain fundamental curves of the geometry involved give the trajectories of charged mass-points in the combined gravitational and electromagnetic field.

The type of geometry employed usually determines the equations of a class of curves that are of special significance for this geometry; and, conversely, if the equations of the trajectories of particles are

† Kaluza, *Sitzungsb. der Preussischen Ak.* (Berlin) (1921), 966; Klein, *Zeits. für Physik*, 37 (1926), 895, and 46 (1927), 188.

‡ Veblen, *Quart. J. of Math.* (Oxford), 1 (1930), 60; Veblen and Hoffmann, *Phys. Rev.* 36 (1930), 810. We shall refer to the latter as P.R.

§ Einstein and Mayer, *Sitzungsb. der Preussischen Ak.* (Berlin) 25 (1931), 541.

‖ Schouten and van Danzig, *Zeits. für Physik*, 78 (1932), 639, and other papers; Veblen, 'Projektive Relativitätstheorie', *Ergebn. der math.* Wiss. (Berlin) (1933), Chap. 8; Hoffmann, *Phys. Rev.* 43 (1933), 615.

known, it is often possible to deduce something concerning the general type of geometry for which these trajectories can be fundamental curves, and also something concerning the values of the fundamental connexion employed in this geometry. Once the geometry is completely specified, the number of possible sets of field equations is often strictly limited. For example, in those field theories such as the projective theory and the present one which closely parallel the general theory of relativity, the field equations are uniquely determined by the requirements that, for free space, they shall be tensor equations of the second rank, they shall be differential equations of the second order in the components of the fundamental tensor, and their generalized divergence shall vanish identically. The last condition ensures that, when a material tensor is included, its generalized divergence will also be zero, thus leading to the laws of change of momentum and energy under the influence of electromagnetic force, and also to the equations of the trajectories of test bodies with which the theory started.† Furthermore, the vanishing of the divergence implies that the field equations are the conditions that some invariant integral shall be an extremum. Thus in favourable cases a definite field theory can be built around the equations expressing the motion, in the type of field one decides to take as fundamental, of the type of particle one decides to take as fundamental.

When the Kaluza-Klein theory and its four-dimensional modifications are viewed from the point of view of the physical significance of their fundamental curves, it is seen that a generalization is necessary; for these fundamental curves make no reference to the motion of test bodies having any magnetic properties. One of the fundamental particles of physics is the electron, which has, in addition to its mass and charge, a magnetic moment that is proportional to its specific charge; its motion, therefore, cannot be correctly described by the fundamental curves of the Kaluza-Klein geometry and its generalizations.

In the present paper the motion of magnetic particles is considered along the lines just discussed. The treatment of the motion of a dipole by this method meets many difficulties, and we shall confine ourselves to the construction of a theory which will take

† For the manner in which this comes about see Einstein and Mayer, *Sitzungsb. der Preussischen Ak.* (Berlin) 25 (1931), 541, § 8.

account of the motion of a test body possessing, in addition to its mass and electric charge, a magnetic pole strength. Such a particle has not been observed in nature, but all magnetic particles can be regarded as formed from a suitable distribution of such magnetic poles, and the situation is equivalent to that which arises in the projective theory of relativity in regard to the motion of test bodies in the form of electric dipoles.

In this paper I develop the theory as a generalization of the Kaluza-Klein theory, and in a later paper I shall show how it can be made into a four-dimensional theory when treated as a generalization of the Einstein-Mayer theory. The present theory has the same degree of freedom as the generalized conformal geometry, and possibly can be brought within the structure of this geometry as a four-dimensional theory, but I have not yet succeeded in accomplishing this.

The theory here presented provides a necessary generalization of the Kaluza-Klein theory and its modifications, and it turns out that in so doing it gives a greater elegance and symmetry to the equations of the electromagnetic field without spoiling their significance.

2. Trajectories

We shall follow the same conventions regarding suffixes as are used by Veblen in conformal geometry.† Thus Latin suffixes will take the values 1, 2, 3, 4, Greek suffixes from the early part of the alphabet will take the values 0, 1, 2, 3, 4, and Greek letters from the later parts of the alphabet will take the values 0, 1, 2, 3, 4, ∞.

For a particle of charge ϵ and mass m the trajectory in a gravitational and electromagnetic field is given by

$$\frac{d^2x^a}{ds^2} + \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} \frac{dx^b}{ds} \frac{dx^c}{ds} = \frac{\epsilon}{m} \phi_b^a \frac{dx^b}{ds}, \tag{1}$$

where g_{ab} and ϕ_{ab} satisfy the field equations of Maxwell and Einstein. These equations may be obtained from the equations of the geodesics of the Kaluza-Klein theory in the following manner. The metric $\gamma_{\alpha\beta}$ has the form‡

$$\left.\begin{aligned} \gamma_{ab} &= g_{ab} + \phi_a \phi_b \\ \gamma_{a0} &= \phi_a \\ \gamma_{00} &= 1 \end{aligned}\right\}, \tag{2}$$

† *Proc. National Ac. Sc.* 21 (1935), 168.

‡ We are using the notation of P.R.

where all the quantities are independent of x^0, and the Christoffel symbols have the values

$$\left.\begin{aligned}
\Gamma^a_{bc} &= \left\{\begin{matrix} a \\ bc \end{matrix}\right\} + \phi^a_b \phi_c + \phi^a_c \phi_b \\
\Gamma^0_{bc} &= -\phi_a \Gamma^a_{bc} + \frac{1}{2}\left(\frac{\partial \phi_b}{\partial x^c} + \frac{\partial \phi_c}{dx^b}\right) \\
\Gamma^a_{0b} &= \Gamma^a_{b0} = \phi^a_b \\
\Gamma^\alpha_{00} &= 0
\end{aligned}\right\}. \tag{3}$$

The geodesics of the Kaluza-Klein space are given by

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0. \tag{4}$$

These equations may be broken up into the two sets

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0,$$

$$\frac{d^2x^0}{ds^2} + \Gamma^0_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0.$$

The former is easily expanded in terms of the values (3) of the Christoffel symbols; the latter may also be expanded by means of (3), and, in the result, we may make use of the first of the above pair of equations to eliminate the quantity $\Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds}$. The two sets of equations then take the forms

$$\left.\begin{aligned}
\frac{d^2x^a}{ds^2} + \left\{\begin{matrix} a \\ bc \end{matrix}\right\}\frac{dx^b}{ds}\frac{dx^c}{ds} + 2\phi^a_b\left(\phi_\gamma\frac{dx^\gamma}{ds}\right)\frac{dx^b}{ds} &= 0 \\
\frac{d^2x^0}{ds^2} + \phi_a\frac{d^2x^a}{ds^2} + \frac{\partial \phi_a}{\partial x^b}\frac{dx^a}{ds}\frac{dx^b}{ds} &= 0
\end{aligned}\right\}. \tag{5}$$

The second equation may be written as

$$\frac{d}{ds}\left(\phi_\gamma\frac{dx^\gamma}{ds}\right) = 0,$$

which shows that $(\phi_\gamma dx^\gamma/ds)$ is constant along the geodesic. If we denote this constant by $(-\epsilon/2m)$, the first equation of (5) reduces to (1).

It is to be noted that the constant $\epsilon/2m$ enters because of the extra dimension x^0, the quantity $(\phi_\gamma dx^\gamma/ds)$ measuring the cosine of the angle between the geodesic under consideration and the x^0-axis. This quantity we should expect to be constant, because of the

'cylinder condition', which postulates that the coordinate x^0 does not enter any geometrical objects of the space.

Now the Lorentz force on a moving electric pole of charge ϵ is $\epsilon(E+v\times H)$, which, in general relativity, becomes $\epsilon\phi_b^a\,dx^b/ds$. The force on a moving magnetic pole of strength μ is $\mu(H+v\times E)$, which becomes $\mu\psi_b^a\,dx^b/ds$, where, in the Galilean case,

$$\psi^{23}=\phi_{14},\ \psi^{31}=\phi_{24},\ \psi^{12}=\phi_{34};\quad \psi^{14}=\phi_{23},\ \psi^{24}=\phi_{31},\ \psi^{34}=\phi_{12},$$

account being taken of changes of sign introduced by raising and lowering indices. Thus, if the test particle has a magnetic pole of strength μ as well as a charge ϵ and a mass m, its trajectory in a gravitational field is given by

$$\frac{d^2x^a}{ds^2}+\left\{\begin{matrix}a\\bc\end{matrix}\right\}\frac{dx^b}{ds}\frac{dx^c}{ds}-\frac{\epsilon}{m}\phi_b^a\frac{dx^b}{ds}-\frac{\mu}{m}\psi_b^a\frac{dx^b}{ds}=0, \tag{6}$$

where ψ_{ab} is the dual of ϕ_{ab}, i.e. where

$$\psi^{ab}=\phi^{*ab}\equiv\frac{1}{2}\frac{1}{\sqrt{(-g)}}\epsilon^{abcd}\phi_{cd}. \tag{7}$$

We now wish to obtain equations (6) as the equations of the fundamental curves of some type of space. Since two constants ϵ/m and μ/m enter the trajectories (6) that we wish to obtain, it is evident that we shall need an extra dimension x^∞ making six dimensions in all. The following scheme gives the trajectories we require.

We consider a space of six dimensions of which four are ordinary space-time while the other two, whose coordinates will be denoted by x^0 and x^∞, satisfy the cylinder condition that the components of all geometrical objects are independent of them. We shall keep x^0 and x^∞ fixed but shall let x^1, x^2, x^3, x^4 undergo the usual general analytical transformations of the space-time frame. The six-dimensional symmetric metrical tensor will be denoted by $\kappa_{\lambda\mu}$, and its associate $\kappa^{\lambda\mu}$ will be defined in the usual manner. From the form of the transformations allowed it is seen that κ^{lm} is a four-dimensional tensor, κ_{l0}, $\kappa_{l\infty}$ are four-dimensional vectors, and κ_{00}, $\kappa_{0\infty}$, $\kappa_{\infty\infty}$ are scalars. Thus if we write

$$\left.\begin{aligned}\kappa^{mn}=g^{mn},\quad \kappa_{\lambda 0}=\phi_\lambda,\quad \kappa_{\lambda\infty}=\psi_\lambda,\quad \kappa_{00}=\phi_0=1,\\ \kappa_{\infty\infty}=\psi_\infty=-1,\quad \kappa_{0\infty}=\phi_\infty=\psi_0=0,\end{aligned}\right\} \tag{8}$$

where g^{mn} is a tensor and ϕ_m and ψ_m are vectors, these equations will be invariant.

Since $\kappa^{\lambda\mu}\kappa_{\mu\nu} = \delta^{\lambda}_{\nu}$, we have

$$\kappa^{lm}\kappa_{m0}+\kappa^{l0}\kappa_{00}+\kappa^{l\infty}\kappa_{\infty 0} = 0,$$

whence $$\kappa^{l0} = -g^{lm}\phi_m. \tag{9}$$

Similarly, $$\kappa^{l\infty} = +g^{lm}\psi_m. \tag{10}$$

Also $$\kappa^{lm}\kappa_{mn}+\kappa^{l0}\kappa_{0n}+\kappa^{l\infty}\kappa_{\infty n} = \delta^{l}_{n},$$

which, in view of (8), (9), (10), may be written as

$$g^{lm}(\kappa_{mn}-\phi_m\phi_n+\psi_m\psi_n) = \delta^{l}_{n},$$

giving $$\kappa_{mn} = g_{mn}+\phi_m\phi_n-\psi_m\psi_n, \tag{11}$$

where $$g^{lm}g_{mn} = \delta^{l}_{n}. \tag{12}$$

Again, since $$\kappa^{0\lambda}\kappa_{0\lambda} = 1,$$

we have $$\kappa^{0l}\phi_l+\kappa^{00} = 1,$$

or, by (9), $$\kappa^{00} = 1+g^{mn}\phi_m\phi_n. \tag{13}$$

Similarly, $$\kappa^{\infty\infty} = -1+g^{mn}\psi_m\psi_n. \tag{14}$$

Finally we have $$\kappa^{0\lambda}\kappa_{\infty\lambda} = 0,$$

i.e. $$\kappa^{0l}\kappa_{\infty l}+\kappa^{00}\kappa_{\infty 0}+\kappa^{0\infty}\kappa_{\infty\infty} = 0,$$

which, by (8) and (10), shows that

$$\kappa^{0\infty} = -g^{mn}\phi_m\psi_n. \tag{15}$$

Thus, collecting results, we have the following values for the components of the metric and its associate:

$$\left.\begin{aligned} &\kappa_{mn} = g_{mn}+\phi_m\phi_n-\psi_m\psi_n, \quad \kappa_{m0} = \phi_m, \\ &\kappa_{m\infty} = \psi_m, \quad \kappa_{0\infty} = 0, \quad \kappa_{00} = 1, \quad \kappa_{\infty\infty} = -1 \end{aligned}\right\}; \tag{16}$$

$$\left.\begin{aligned} &\kappa^{mn} = g^{mn}, \quad \kappa^{m0} = -g^{mn}\phi_n, \quad \kappa^{m\infty} = g^{mn}\psi_n, \\ &\kappa^{0\infty} = -g^{mn}\phi_m\psi_n, \quad \kappa^{00} = 1+g^{mn}\phi_m\phi_n, \quad \kappa^{\infty\infty} = -1+g^{mn}\psi_m\psi_n \end{aligned}\right\}. \tag{17}$$

From (16) we easily obtain the result that the determinant $\kappa \equiv |\kappa_{\lambda\mu}|$ is equal to the negative of the determinant $g \equiv |g_{mn}|$.

We now compute the values of the Christoffel symbols $K^{\lambda}_{\mu\nu}$ formed from the $\kappa_{\lambda\mu}$ in the usual manner:

$$K^{\lambda}_{\mu\nu} \equiv \tfrac{1}{2}\kappa^{\lambda\sigma}\left(\frac{\partial\kappa_{\sigma\nu}}{\partial x^{\mu}}+\frac{\partial\kappa_{\sigma\mu}}{\partial x^{\nu}}-\frac{\partial\kappa_{\mu\nu}}{\partial x^{\sigma}}\right).$$

If we denote by $\left\{ \begin{matrix} l \\ mn \end{matrix} \right\}$ the Christoffel symbols formed from the g_{mn}, by ϕ_{mn} the tensor $\frac{1}{2}\left(\frac{\partial\phi_m}{dx^n}-\frac{\partial\phi_n}{dx^m}\right)$, and by ψ_{mn} the tensor $\frac{1}{2}\left(\frac{\partial\psi_m}{\partial x^n}-\frac{\partial\psi_n}{\partial x^m}\right)$, and if we use the g's for raising and lowering suffixes of ϕ_{mn} and ψ_{mn}

and follow the convention that the first suffix will be raised first, we find that the various components of the connexion $K^{\lambda}_{\mu\nu}$ take on the values:

$$\left.\begin{aligned}
K^{l}_{mn} &= \left\{ \begin{matrix} l \\ mn \end{matrix} \right\} + \phi^{l}_{m}\phi_{n} + \phi^{l}_{n}\phi_{m} - \psi^{l}_{m}\psi_{n} - \psi^{l}_{n}\psi_{m} \\
K^{0}_{mn} &= -\phi_{l}K^{l}_{mn} + \frac{1}{2}\left(\frac{\partial\phi_{m}}{\partial x^{n}} + \frac{\partial\phi_{n}}{\partial x^{m}}\right) \\
K^{\infty}_{mn} &= \psi_{l}K^{l}_{mn} - \frac{1}{2}\left(\frac{\partial\psi_{m}}{\partial x^{n}} + \frac{\partial\psi_{n}}{\partial x^{m}}\right) \\
K^{n}_{m0} &= \phi^{n}_{m}, \quad K^{n}_{m\infty} = \psi^{n}_{m}, \quad K^{0}_{m0} = -\phi^{n}_{m}\phi_{n}, \quad K^{\infty}_{m\infty} = \psi^{n}_{m}\psi_{n} \\
K^{0}_{m\infty} &= -\psi^{n}_{m}\phi_{n}, \quad K^{\infty}_{m0} = \phi^{n}_{m}\psi_{n}, \quad K^{\lambda}_{0\infty} = K^{\lambda}_{00} = K^{\lambda}_{\infty\infty} = 0
\end{aligned}\right\} \quad (18)$$

We now assume that the trajectories are given by the geodesics

$$\frac{d^{2}x^{\lambda}}{ds^{2}} + K^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0. \quad (19)$$

These equations may be separated into three groups, those for which $\lambda = 1, 2, 3, 4$, that for which $\lambda = 0$, and that for which $\lambda = \infty$. For $\lambda = l$ we find that the corresponding equations may be written as

$$\frac{d^{2}x^{l}}{ds^{2}} + \left\{ \begin{matrix} l \\ mn \end{matrix} \right\}\frac{dx^{m}}{ds}\frac{dx^{n}}{ds} + 2\left(\phi_{\nu}\frac{dx^{\nu}}{ds}\right)\phi^{l}_{m}\frac{dx^{m}}{ds} + 2\left(\psi_{\nu}\frac{dx^{\nu}}{ds}\right)\psi^{l}_{m}\frac{dx^{m}}{ds} = 0. \quad (20)$$

As in § 2, we may now use the equations (20) to eliminate $K^{l}_{mn}\frac{dx^{m}}{ds}\frac{dx^{n}}{ds}$ from the equations for $\lambda = 0$ and $\lambda = \infty$. After this elimination has been performed, the remaining equations take the forms:

for $\lambda = 0$, $$\frac{d^{2}x^{0}}{ds^{2}} + \phi_{m}\frac{d^{2}x^{m}}{ds^{2}} + \frac{\partial\phi_{m}}{\partial x^{n}}\frac{dx^{m}}{ds}\frac{dx^{n}}{ds} = 0; \quad (21)$$

and for $\lambda = \infty$,

$$-\frac{d^{2}x^{\infty}}{ds^{2}} + \psi_{m}\frac{d^{2}x^{m}}{ds^{2}} + \frac{\partial\psi_{m}}{\partial x^{n}}\frac{dx^{m}}{ds}\frac{dx^{n}}{ds} = 0. \quad (22)$$

Equation (21) may be written as

$$\frac{d}{ds}\left(\phi_{\lambda}\frac{dx^{\lambda}}{ds}\right) = 0,$$

which shows that, along a given geodesic, the quantity $\left(\phi_{\lambda}\frac{dx^{\lambda}}{ds}\right)$ is constant. We denote this constant by $(-\epsilon/2m)$. Similarly, equation (22) shows that, along a given geodesic, the quantity $\left(\psi_{\lambda}\frac{dx^{\lambda}}{ds}\right)$ is constant. We denote this constant by $(-\mu/2m)$.

The equations (20) may now be written as

$$\frac{d^2x^l}{ds^2}+\left\{\begin{matrix} l \\ mn \end{matrix}\right\}\frac{dx^m}{ds}\frac{dx^n}{ds}-\frac{\epsilon}{m}\phi^l_m\frac{dx^m}{ds}-\frac{\mu}{m}\psi^l_m\frac{dx^m}{ds}=0, \tag{23}$$

and these will be precisely the equations (6) provided we relate ϕ_{mn} and ψ_{mn} by relation (7).

Thus we shall obtain the trajectories we require provided we take the metric $\kappa_{\lambda\mu}$ such that ϕ_m and ψ_m satisfy the relations

$$g^{mp}g^{nq}\left(\frac{\partial\psi_p}{\partial x^q}-\frac{\partial\psi_q}{\partial x^p}\right)=\frac{\epsilon^{mnpq}}{2\sqrt{(-g)}}\left(\frac{\partial\phi_p}{\partial x^q}-\frac{\partial\phi_q}{\partial x^p}\right). \tag{24}$$

We shall discuss the integrability of these conditions in § 4.

3. Field equations

Field equations may be constructed in the same manner as in the projective theory. These field equations will naturally place restrictions upon ϕ_m and ψ_m, but it turns out that they are completely compatible with the restriction (24) that has already been imposed on these quantities.

In order to construct the field equations, we first form the six-dimensional Ricci tensor $C_{\lambda\mu}$ given by

$$C_{\lambda\mu}=\frac{\partial K^\sigma_{\sigma\lambda}}{\partial x^\mu}-\frac{\partial K^\sigma_{\lambda\mu}}{\partial x^\sigma}+K^\tau_{\lambda\sigma}K^\sigma_{\tau\mu}-K^\tau_{\mu\lambda}K^\sigma_{\sigma\tau}. \tag{29}$$

It is found that this tensor has the following components:

$$\left.\begin{aligned} C_{mn} &= R_{mn}-\phi_n\phi^s_{m,s}-\phi_m\phi^s_{n,s}+\psi_n\psi^s_{m,s}+\psi_m\psi^s_{n,s}-2\phi^s_m\phi_{ns}+ \\ &\quad +2\psi^s_m\psi_{ns}+\phi_m\phi_n\phi^t_s\phi^s_t-\phi_m\psi_n\phi^t_s\psi^s_t- \\ &\quad -\psi_m\phi_n\psi^t_s\phi^s_t+\psi_m\psi_n\psi^t_s\psi^s_t \\ C_{m0} &= -\phi^s_{m,s}+\phi_m\phi^s_t\phi^t_s-\psi_m\phi^s_t\psi^t_s \\ C_{m\infty} &= -\psi^s_{m,s}-\psi_m\psi^s_t\psi^t_s+\phi_m\psi^s_t\phi^t_s \\ C_{00} &= \phi^s_t\phi^t_s, \quad C_{\infty\infty}=\psi^s_t\psi^t_s, \quad C_{0\infty}=\psi^s_t\phi^t_s \end{aligned}\right\}, \tag{30}$$

where the commas denote covariant differentiation with respect to the g_{mn}, and R_{mn} is the Ricci tensor formed out of the g_{mn}. From these results it follows at once that the curvature scalar C has the value

$$C\equiv\kappa^{\lambda\mu}C_{\lambda\mu}=R-\phi^s_t\phi^t_s+\psi^s_t\psi^t_s, \tag{31}$$

where R is the curvature scalar formed from the g_{mn}.

The field equations will be taken as the conditions that

$$\delta\int C\sqrt{(-g)}\,dx^1dx^2dx^3dx^4=0, \tag{32}$$

where the variation is with respect to the $\kappa_{\lambda\mu}$, the variations $\delta\kappa_{\lambda\mu}$ and their first and second derivatives vanishing at the boundaries. The above variational principle may be written as†

$$\int \Omega^{\lambda\mu}\delta\kappa_{\lambda\mu}\sqrt{(-g)}\,dx^1dx^2dx^3dx^4 = 0, \tag{33}$$

where

$$\Omega^{\lambda\mu} \equiv C^{\lambda\mu} - \tfrac{1}{2}\kappa^{\lambda\mu}C. \tag{34}$$

Since $\kappa_{00} = 1$, $\kappa_{0\infty} = 0$, and $\kappa_{\infty\infty} = -1$, the variations $\delta\kappa_{\lambda\mu}$ are not independent. We have always $\delta\kappa_{00} = 0$, $\delta\kappa_{0\infty} = 0$, $\delta\kappa_{\infty\infty} = 0$, so that from (33) we can only obtain

$$\Omega^{\lambda\mu} + U\delta^\lambda_0\,\delta^\mu_0 + V(\delta^\lambda_0\,\delta^\mu_\infty + \delta^\lambda_\infty\,\delta^\mu_0) + W\delta^\lambda_\infty\,\delta^\mu_\infty = 0,$$

where U, V, W are undetermined functions.

If we lower the indices we obtain

$$\Omega_{\lambda\mu} + U\phi_\lambda\phi_\mu + V(\phi_\lambda\psi_\mu + \psi_\lambda\phi_\mu) + W\psi_\lambda\psi_\mu = 0.$$

For $\lambda = \mu = 0$, these give

$$U = -\Omega_{00};$$

similarly, for $\lambda = 0$, $\mu = \infty$, and for $\lambda = \infty$, $\mu = \infty$, we obtain respectively

$$V = -\Omega_{0\infty}, \qquad W = -\Omega_{\infty\infty}.$$

The field equations may therefore be written as‡

$$T_{\lambda\mu} \equiv \Omega_{\lambda\mu} - \phi_\lambda\phi_\mu\Omega_{00} - (\phi_\lambda\psi_\mu + \psi_\lambda\phi_\mu)\Omega_{0\infty} - \psi_\lambda\psi_\mu\Omega_{\infty\infty} = 0. \tag{35}$$

The equations

$$T_{00} = 0, \qquad T_{0\infty} = 0, \qquad T_{\infty 0} = 0, \qquad T_{\infty\infty} = 0$$

are obviously identically satisfied. The remaining equations may be regrouped as

$$T^{mn} = 0, \tag{36}$$

$$T^m_0 = 0, \tag{37}$$

$$T^m_\infty = 0. \tag{38}$$

† The method is the same as that of P.R. 817–18.

‡ In P.R. the field equations were $\Gamma_{\alpha\beta} - \phi_\alpha\phi_\beta\Gamma = 0$, whereas the analogue of the field equations of the present paper would be $\Gamma_{\alpha\beta} - \phi_\alpha\phi_\beta\Gamma_{00} = 0$. There is no essential difference between these two sets of equations, however, since each contains only fourteen algebraically independent equations, the second set quite obviously so, since the '0−0' equation is identically satisfied. The difference lies only in the fact that the equations of the first set are the result of taking linear combinations of the field equations of the second set. It is evident that other, more complicated, expressions could be used to give the same field equations, but the field equations, whether considered as the conditions that a certain integral be extremal, or as equations governing the physical field, have a unique significance. Similar remarks, of course, apply to the field equations of the present paper.

By an obvious generalization of a theorem I have proved elsewhere,† we see from the form of the scalar C that the field equations obtained from (32) must, when grouped as in (36), (37), (38), take on the respective forms

$$(R^{mn}-\tfrac{1}{2}g^{mn}R)+2(g^{st}\phi_s^m\phi_t^n+\tfrac{1}{4}g^{mn}\phi_t^s\phi_s^t)- \\ -2(g^{st}\psi_s^m\psi_t^n+\tfrac{1}{4}g^{mn}\psi_t^s\psi_s^t)=0, \qquad (36')$$

$$\phi^{ms}{}_{,s}=0, \qquad (37')$$

$$\psi^{ms}{}_{,s}=0; \qquad (38')$$

and it affords a useful test of the accuracy of our computations to verify that the equations (36), (37), (38) do in fact take the forms (36′), (37′), (38′), respectively.

When we take into account the conditions (24), we see that the two sets of equations, (37) and (38), are the complete set of Maxwell equations expressed in symmetric form. We shall discuss these in the next section.

It is not difficult to show that the remaining field equations (36) now reduce essentially to the corresponding field equations of the projective theory. This is most simply seen by considering the value of the scalar C that enters the variational principle (32) when the conditions (24) are imposed. In the usual treatment of the relative tensors ϵ^{abcd} and ϵ_{abcd}, the quantity $\sqrt{g}$ is used in order to reduce them to absolute tensors; but in relativity it is necessary to use $\sqrt{(-g)}$ if we wish to avoid complex quantities. The effect of this is not negligible in the present instance, for if we evaluate $\psi^{st}\psi_{st}$ in terms of ϕ_{st}, we obtain

$$\begin{aligned}\psi^{st}\psi_{st}=g_{sp}g_{tq}\psi^{st}\psi^{pq}&=g_{sp}g_{tq}\phi^{*st}\phi^{*pq}\\&=\frac{1}{4(-g)}g_{sp}g_{tq}\epsilon^{stab}\epsilon^{pqcd}\phi_{ab}\phi_{cd}\\&=\frac{1}{4(-g)}(g_{sp}g_{tq}g_{am}g_{bn}\epsilon^{stab})\epsilon^{pqcd}\phi^{mn}\phi_{cd}\\&=\frac{1}{4(-g)}(+g)\epsilon_{pqmn}\epsilon^{pqcd}\phi^{mn}\phi_{cd}\\&=-\phi^{cd}\phi_{cd}=-\phi^{st}\phi_{st}.\end{aligned}$$

Thus the use of $\sqrt{(-g)}$ has introduced a negative sign in the value of $\phi^{*st}\phi^*_{st}$ that would not have appeared had we been able to use $\sqrt{g}$.

† Hoffmann, *Proc. Royal Soc.* A 148 (1935), 353. Lemma.

Since $\psi^{st}\psi_{st}$ is the negative of $\phi^{st}\phi_{st}$, the variational principle (32) reduces, for variations of the g_{mn} alone, to

$$\delta \int (R-2\phi^{st}\phi_{st})\sqrt{(-g)}\, dx^1 dx^2 dx^3 dx^4 = 0, \tag{39}$$

which will therefore give for the field equations (36) the form

$$(R^{mn}-\tfrac{1}{2}g^{mn}R)+4(g^{st}\phi_s^m\phi_t^n+\tfrac{1}{4}g^{mn}\phi_t^s\phi_s^t)=0; \tag{36''}$$

this can be verified by direct computation from the form of (36'). The above equations, except for the change of units implied by the 4-factor that takes the place of the 2-factor of the corresponding field equations of the projective theory, are identical with the field equations of that theory.

Thus we have shown that the field equations of the present theory do not contradict the results of the projective theory.

(The negative sign introduced by $\sqrt{(-g)}$ is the reason for taking the metric $\kappa_{\lambda\mu}$ to be such that κ_{00} and $\kappa_{\infty\infty}$ have opposite signs. If κ_{00} and $\kappa_{\infty\infty}$ were taken to be of the same sign, the scalar C would take the form $(R-\phi_t^s\phi_s^t-\psi_t^s\psi_s^t)$ and the electromagnetic part of the energy would thus vanish when ϕ^{*st} was identified with ψ^{st}.)

4. Field equations and the integrability of (24)

In this section we shall discuss the relation (24) in the light of the field equations. Let us, for the moment, forget that ϕ_{mn} and ψ_{mn} are the four-dimensional curls of vectors ϕ_m and ψ_m. The field equations (37) can be replaced by the equivalent invariant equations

$$\frac{\partial\phi_{mn}^*}{\partial x^p}+\frac{\partial\phi_{np}^*}{\partial x^m}+\frac{\partial\phi_{pm}^*}{\partial x^n}=0, \tag{40}$$

where ϕ_{mn}^*, the covariant dual of ϕ_{mn}, is defined by

$$\phi_{mn}^* = g_{ms}g_{nt}\phi^{*st} = -\tfrac{1}{2}\sqrt{(-g)}\epsilon_{mnpq}\phi^{pq}. \tag{41}$$

The equations (40) are the conditions that we shall be able to find a vector, say ϕ_m^*, such that

$$\phi_{mn}^* = -\frac{1}{2}\left(\frac{\partial\phi_m^*}{\partial x^n}-\frac{\partial\phi_n^*}{\partial x^m}\right). \tag{42}$$

Similarly, if the field equations (38) are satisfied, we can find a ψ_m^* such that

$$\psi_{mn}^* = \frac{1}{2}\left(\frac{\partial\psi_m^*}{\partial x^n}-\frac{\partial\psi_n^*}{\partial x^m}\right). \tag{43}$$

If, therefore, we identify ϕ_m^* with ψ_m and ψ_m^* with ϕ_m, we shall have

$$\phi_{mn}^* = -\psi_{mn}, \qquad \psi_{mn}^* = +\phi_{mn}.$$

These are consequences of each other and are the same as the single relation

$$\psi^{mn} = \phi^{*mn}; \tag{44}$$

and when (42), (43), and the relationship between ϕ^*_m, ψ^*_m, ϕ_m, ψ_m are taken into account, (44) reduces precisely to (24).

Thus the field equations (37) and (38), which are merely the Maxwell equations, ensure that the conditions (24) will be integrable.

Since ϕ_{mn} is the curl of a vector, the field equations

$$\phi^{*mn}{}_{,n} = 0 \tag{38''}$$

will be identically satisfied; and a similar result holds for ψ. It is seen, therefore, that there is a certain redundance about the conditions we have imposed, since, in building the present theory from the metric $\kappa_{\lambda\mu}$ in such a way that ϕ_{mn} and ψ_{mn} are automatically the curls of vectors, we have anticipated part of what can be deduced merely from the field equations (37) and (38) in conjunction with the postulate that ϕ_{mn} and ψ_{mn} are dual antisymmetric tensors. If we ignore the fact that ϕ_{mn} and ψ_{mn} are curls of vectors, but assume that they are dual six-vectors, the two sets of field equations (37) and (38) will constitute the complete set of Maxwell equations and will include the fact that ϕ_{mn} and its dual are curls of vectors. Either (37) or (38) alone gives the complete set of Maxwell equations when we take into account respectively the facts that ϕ_{mn} and ψ_{mn} are curls of vectors.

There is no room, physically, for two electromagnetic fields and we could, therefore, have no use for a ψ_{mn} that bore no relation at all to ϕ_{mn}. The identification of ϕ_{mn} with ψ_{mn} would have removed the unwanted freedom, but the present identification of ψ_{mn} with the dual of ϕ_{mn} has accomplished the same result, not only with greater mathematical elegance, but also with increased physical significance.

A GENERALIZATION OF THE EINSTEIN-MAYER FIELD THEORY

By B. HOFFMANN (*Princeton*)

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I

1. Introduction

In the previous paper* a generalization of the Kaluza-Klein field theory was proposed in order that the motion of test particles having magnetic properties could be treated in the same manner as one treats the motion of test particles having electrical properties. The generalization determined by the physical argument turned out to involve a six-dimensional geometry, and, though this seemed to bear some relation to the conformal geometry of four dimensions, no explicit relationship was found. Since the introduction of dimensions to which no physical significance has been assigned is open to objection, I discuss in the present paper a four-dimensional theory constructed in analogy with the theory of Einstein and Mayer,† which contains all the results of the six-dimensional theory, and is, in fact, intimately related to that theory through a non-holonomic transformation.

In the first part I shall develop the four-dimensional theory in the manner of the theory of Einstein and Mayer, and later I shall show the relation between the two theories. The present generalization of the Einstein-Mayer theory not only retains all the advantages of the generalization given in the previous paper, while remaining four-dimensional, but it also finds a natural place for the awkward set of field equations, E.M. (47), of the Einstein-Mayer theory. The form of the present work, however, seems more remote from the conformal geometry than that of the work of the previous paper.

2. Four-Dimensional and Six-Dimensional Vectors

Since the method of the present paper follows closely that of the Einstein-Mayer paper, I shall omit many details that would be necessary for a complete exposition of the subject.

* Quoted as G.K.K.

† Einstein and Mayer, *Sitzungsb. der Preussischen Ak.* (Berlin) 25 (1931), 541, quoted as E.M.

With every point (x^m) of the underlying space-time R_4 is associated a linear six-dimensional vector space V_6 having a symmetric metrical tensor* $\kappa_{\lambda\mu}$. The metric of the underlying space is given by the tensor g_{mn}. Suffixes of vectors in V_6 are raised and lowered by means of $\kappa_{\lambda\mu}$, in R_4 by means of g_{mn}. Indices of mixed tensors are raised and lowered by means of the $\kappa_{\lambda\mu}$ and the g_{mn} according as they belong respectively to V_6 or to R_4. Tensors in V_6 and R_4 are associated through a mixed tensor γ_λ^k by means of formulae such as

$$a^k = \gamma_\lambda^k a^\lambda, \tag{1}$$

$$b_\lambda = \gamma_\lambda^k b_k, \tag{2}$$

and indices of γ_λ^k are, in accordance with the general rule, raised and lowered by means of the $\kappa_{\lambda\mu}$ and the g_{mn} respectively, according as the index in question belongs to V_6 or to R_4.

We assume that the rank of the matrix $||\gamma_\lambda^k||$ is four. Then, if v_k is an arbitrary vector of R_4, the vector $\gamma_\lambda^k v_k$ spans a four-dimensional sub-space of V_4. The two-dimensional normal sub-space S_2 will be spanned by two mutually perpendicular unit vectors, say A^λ and B^λ. Since A^λ is perpendicular to any $\gamma_\lambda^k v_k$, where v_k is a vector of space-time, we must have

$$A^\lambda \gamma_\lambda^k v_k = 0,$$

for all v_k. Hence we must have

$$A^\lambda \gamma_\lambda^k = 0. \tag{3}$$

Similarly,

$$B^\lambda \gamma_\lambda^k = 0. \tag{4}$$

Since A^λ and B^λ have been taken as orthogonal, we shall have

$$A^\lambda B_\lambda = 0, \tag{5}$$

and, since the two vectors are unit vectors, we may take

$$A^\mu A_\mu \equiv \kappa_{\lambda\mu} A^\lambda A^\mu = 1, \tag{6}$$

$$B^\mu B_\mu \equiv \kappa_{\lambda\mu} B^\lambda B^\mu = -1. \tag{7}$$

The relations (5), (6), (7) help to determine the metric $\kappa_{\lambda\mu}$ in terms of A_λ and B_λ. The negative sign in (7) implies that the metric $\kappa_{\lambda\mu}$ has the signature -2 instead of zero; its ultimate effect is to introduce the necessary negative sign before the third parenthesis of the field equations (40).

We now compute the quantity

$$\Sigma_\lambda^\mu \equiv \gamma_\lambda^p \gamma_p^\mu. \tag{8}$$

* We use the same conventions for suffixes as in G.K.K., § 2.

Multiplying by γ_q^λ, we find

$$\gamma_q^\lambda(\delta_\lambda^\mu - \Sigma_\lambda^\mu) = 0.$$

Hence, by (3) and (4), we must have

$$\delta_\lambda^\mu - \Sigma_\lambda^\mu = C^\mu A_\lambda + D^\mu B_\lambda, \tag{9}$$

where C^μ and D^μ are vectors of V_6.

From (8), on multiplying by A^λ, we obtain, using (3),

$$A^\lambda \Sigma_\lambda^\mu = 0.$$

Similarly,

$$B^\lambda \Sigma_\lambda^\mu = 0.$$

Therefore, multiplying (9) by A^λ and using (5) and (6), we find that

$$A^\mu = C^\mu.$$

Similarly,

$$B^\mu = -D^\mu.$$

Thus, finally,

$$\gamma_\lambda^p \gamma_p^\mu = \Sigma_\lambda^\mu = \delta_\lambda^\mu - A^\mu A_\lambda + B^\mu B_\lambda. \tag{10}$$

Lowering the index μ, we get

$$g_{pq}\gamma_\lambda^p \gamma_\mu^q = \kappa_{\lambda\mu} - A_\lambda A_\mu + B_\lambda B_\mu,$$

i.e.

$$\kappa_{\lambda\mu} = g_{pq}\gamma_\lambda^p \gamma_\mu^q + A_\lambda A_\mu - B_\lambda B_\mu. \tag{11}$$

3. Covariant Differentiation

We introduce a connexion $\Gamma_{\mu p}^\lambda$ analogous to that of E.M. and employ it in the usual manner for taking covariant derivatives for each index belonging to V_6; for indices belonging to R_4 we use the ordinary Christoffel symbols $\left\{ \begin{matrix} l \\ mn \end{matrix} \right\}$ formed from the g_{mn}. If we denote the covariant derivative of a tensor by means of a solidus, we have, for example,

$$\gamma_{k|m}^\lambda = \frac{\partial \gamma_k^\lambda}{\partial x^m} + \Gamma_{\mu m}^\lambda \gamma_k^\mu - \left\{ \begin{matrix} n \\ km \end{matrix} \right\} \gamma_n^\lambda.$$

All the usual laws of covariant differentiation hold for the present type of covariant derivative.

4. Determination of the $\Gamma_{\mu m}^\lambda$

In order to find definite values for the $\Gamma_{\mu m}^\lambda$, as yet quite general, we must consider properties of parallel displacement that we wish the geometry to possess. We impose the following requirements:

(i) the absolute derivative of $\kappa_{\lambda\mu}$ shall vanish: this is essential if we are to treat a^λ and a_λ on an equal footing in parallel displacement;

(ii) if a vector a^k undergoes parallel displacement, then the absolute differential of $a^\lambda \equiv \gamma^\lambda_k a^k$ shall lie in S_2;

(iii) if, in particular, a^k undergoes parallel displacement in its own direction, then the absolute differential of a^λ shall vanish.

The above three conditions restrict the generality of the geometry; the fourth condition we shall impose is more in the nature of a convention. The vectors A^λ and B^λ are chosen at each point of space-time subject only to the requirements that they shall be unit and orthogonal, and these are really conventions concerning the metric $\kappa_{\lambda\mu}$. Since we have not yet related the A's and B's belonging to different points of space-time, we may make the convention that they shall form a double holonomic field under parallel displacement, which means that their orientations are always to be so chosen that the absolute differential of A^λ shall be perpendicular to B^λ and vice versa.* This is our condition (iv).

Condition (i) may be written

$$\kappa_{\lambda\mu|m} = 0. \tag{12}$$

From condition (ii) we find that the quantity $a^k dx^q \gamma^\lambda_{k|q}$ must lie in S_2. We must therefore have

$$\gamma^\lambda_{k|q} = -A^\lambda F_{kq} + B^\lambda H_{kq}, \tag{13}$$

where F_{kq} and H_{kq} are arbitrary second order tensors belonging to R_4. From condition (iii) we find that

$$0 = \gamma^\lambda_{k|q} a^k a^q = -(A^\lambda F_{kq} - B^\lambda H_{kq}) a^k a^q.$$

If we multiply by A_λ and use (5) and (6), we at once find that

$$F_{kq} a^k a^q = 0,$$

whence

$$F_{kq} = -F_{qk}. \tag{14}$$

Similarly,

$$H_{kq} = -H_{qk}. \tag{15}$$

From condition (iv) we want

$$B^\lambda A_{\lambda|m} = 0, \qquad A^\lambda B_{\lambda|m} = 0. \tag{16}$$

These conditions are equivalent to each other in view of (5) and (12). Also, from (6) and (12), and from (7) and (12), respectively, we find that

$$A^\lambda A_{\lambda|m} = 0, \qquad B^\lambda B_{\lambda|m} = 0. \tag{17}$$

* It is unnecessary to include that the absolute differentials of A^λ and B^λ shall also be respectively perpendicular to A^λ and B^λ, since these follow at once from (6) and (7) in conjunction with condition (i) above.

From (13), using (5) and (6), we obtain

$$A_\lambda \gamma^\lambda_{k|q} = -F_{kq},$$

and since $A_\lambda \gamma^\lambda_k$ is zero, we may write this as

$$F_{kq} = \gamma^\lambda_k A_{\lambda|q}.$$

If we multiply this by γ^k_ν and use (10), we find that

$$\gamma^k_\nu F_{kq} = (\delta^\lambda_\nu - A^\lambda A_\nu + B^\lambda B_\nu) A_{\lambda|q},$$

and, by (16) and (17), this reduces to

$$A_{\nu|q} = \gamma^k_\nu F_{kq}. \tag{18}$$

In a similar manner we may show that

$$B_{\nu|q} = \gamma^k_\nu H_{kq}. \tag{19}$$

We have thus so determined the connexion $\Gamma^\lambda_{\mu m}$ that the covariant derivatives of the fundamental quantities γ^k_λ, A_λ, B_λ, are given by the equations (13), (18), (19), with the conditions (14), (15). A final restriction on $\Gamma^\lambda_{\mu m}$ will be imposed in the next section.

5. Trajectories

We make the condition that the trajectories of test bodies having mass, charge, and magnetic pole strength shall be given by those curves of space-time that are generated by the parallel displacement of a vector a^λ of V_6 in the direction of its associated vector $a^\lambda \gamma^k_\lambda$ of R_4.

The computation is completely analogous to that of E.M. § 4. We may choose a parameter t such that $a^k = \gamma^k_\lambda a^\lambda = dx^k/dt$. Then, since δa^λ is zero, we must have

$$\delta a^k = \gamma^k_{\lambda|p} a^\lambda a^p dt,$$

or, by (12),

$$\frac{d^2x^k}{dt^2} + \begin{Bmatrix} k \\ pq \end{Bmatrix} \frac{dx^p}{dt}\frac{dx^q}{dt} = -(A_\lambda F^k_p - B_\lambda H^k_p) a^\lambda \frac{dx^p}{dt}. \tag{20}$$

We next show that $A_\lambda a^\lambda$ and $B_\lambda a^\lambda$ are constant along the curve. For example, we have

$$\frac{\delta(A_\lambda a^\lambda)}{\delta t} = A_{\lambda|p} a^p a^\lambda + A_\lambda \frac{\delta a^\lambda}{\delta t};$$

the second term on the right vanishes because δa^λ is zero along the curve, and, from (18), the first term may be written as

$$-\gamma^k_\lambda F_{kp} a^p a^\lambda = -F_{kp} a^p a^k,$$

which vanishes because of (14). Therefore $A_\lambda a^\lambda$ is constant along the curve. A similar proof holds for $B_\lambda a^\lambda$.

If we write $(-\epsilon/m)$ for the constant $A_\lambda a^\lambda$, and $(+\mu/m)$ for the constant $B_\lambda a^\lambda$, we may now write (20) as

$$\frac{d^2x^k}{dt^2}+\left\{\begin{matrix}k\\pq\end{matrix}\right\}\frac{dx^p}{dt}\frac{dx^q}{dt}=\frac{\epsilon}{m}F^k_p\frac{dx^p}{dt}+\frac{\mu}{m}H^k_p\frac{dx^p}{dt}. \tag{21}$$

Since, along the curve, δa^λ is zero, we must also have $\delta(\kappa_{\lambda\mu}a^\lambda a^\mu)$ zero. Thus $\kappa_{\lambda\mu}a^\lambda a^\mu$ will be constant, and, by (11), this may be written as

$$g_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}+\frac{\epsilon^2}{m^2}-\frac{\mu^2}{m^2}=\text{constant},$$

i.e.

$$g_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}=\text{constant}.$$

So, if we define ds in the usual manner by

$$ds^2=g_{pq}dx^pdx^q,$$

we have

$$\frac{ds}{dt}=\text{constant},$$

showing that, to within a numerical factor and an additive constant, the parameter t is the same as the interval s. There will therefore be no loss of generality if we take t to be the same as s, and in this case the curves (21) will take the form

$$\frac{d^2x^k}{ds^2}+\left\{\begin{matrix}k\\pq\end{matrix}\right\}\frac{dx^p}{ds}\frac{dx^q}{ds}=\frac{\epsilon}{m}F^k_p\frac{dx^p}{ds}+\frac{\mu}{m}H^k_p\frac{dx^p}{ds}. \tag{22}$$

These will be the trajectories of test particles having mass m, electric charge ϵ, and magnetic pole strength μ, provided that the curve is in a time-like direction; that

$$H^{kp}=F^{*kp}\equiv\epsilon^{kprs}F_{rs}/2\sqrt{(-g)}, \tag{23}$$

apart from an unessential factor of proportionality; and that F^{kp} and g^{kp} satisfy the field equations of Maxwell and Einstein. We shall therefore impose the condition (23) on F_{kp} and H_{kp}; the field equations of the present theory will show that F_{kp} and g_{kp} satisfy the Maxwell-Einstein field equations.

6. Field Equations

It can be shown, by the method of E.M. § 5, that the quantity

$$P^\nu_{\lambda pq}\equiv\frac{\partial\Gamma^\nu_{\lambda p}}{\partial x^q}-\frac{\partial\Gamma^\nu_{\lambda q}}{\partial x^p}+\Gamma^\nu_{\tau p}\Gamma^\tau_{\lambda q}-\Gamma^\nu_{\tau q}\Gamma^\tau_{\lambda p} \tag{24}$$

is a mixed tensor of the type indicated by its suffixes. It is the analogue, in the present geometry, of the usual curvature tensor.

We seek an explicit form for it. From (13), (18), (19), we find that

$$\gamma_{\lambda k|q|p}-\gamma_{\lambda k|p|q} = -A_\lambda(F_{kq|p}-F_{kp|q})+B_\lambda(H_{kq|p}-H_{kp|q})- \\ -\gamma^r_\lambda[(F_{kq}F_{rp}-F_{kp}F_{rq})-(H_{kq}H_{rp}-H_{kp}H_{rq})]; \quad (25)$$

from (18), using (13), we find that

$$A_{\lambda|p|q}-A_{\lambda|q|p} = \gamma^k_\lambda(F_{kp|q}-F_{kq|p})+B_\lambda(F_{kp}H^k_q-F_{kq}H^k_p); \quad (26)$$

and from (19) and (13) we find that

$$B_{\lambda|p|q}-B_{\lambda|q|p} = \gamma^k_\lambda(H_{kp|q}-H_{kq|p})-A_\lambda(H_{kp}F^k_q-H_{kq}F^k_p). \quad (27)$$

But the rules for changing the order of covariant differentiation applicable to the present geometry lead at once to the formulae

$$\gamma_{\lambda k|q|p}-\gamma_{\lambda k|p|q} = -P^\nu_{\lambda qp}\gamma_{\nu k}-R^r_{kqp}\gamma_{\lambda r}, \quad (28)$$

$$A_{\lambda|p|q}-A_{\lambda|q|p} = -P^\nu_{\lambda pq}A_\nu, \quad (29)$$

$$B_{\lambda|p|q}-B_{\lambda|q|p} = -P^\nu_{\lambda pq}B_\nu. \quad (30)$$

Thus, on comparing (25), (26), (27), respectively, with (28), (29), (30), we find that

$$P^\nu_{\lambda qp}\gamma_{\nu k} = A_\lambda(F_{kq|p}-F_{kp|q})-B_\lambda(H_{kq|p}-H_{kp|q})+ \\ +\gamma^r_\lambda[(F_{kq}F_{rp}-F_{kp}F_{rq})-(H_{kq}H_{rp}-H_{kp}H_{rq})-R_{rkqp}], \quad (31)$$

$$P^\nu_{\lambda pq}A_\nu = -\gamma^k_\lambda(F_{kp|q}-F_{kq|p})-B_\lambda(F_{kp}H^k_q-F_{kq}H^k_p), \quad (32)$$

$$P^\nu_{\lambda pq}B_\nu = -\gamma^k_\lambda(H_{kp|q}-H_{kq|p})+A_\lambda(H_{kp}F^k_q-H_{kq}F^k_p). \quad (33)$$

Multiplying (31) by $\gamma^{\tau k}$, and using (10), (32), (33), we obtain

$$P^\tau_{\lambda qp} = \gamma^{\tau k}[A_\lambda(F_{kq|p}-F_{kp|q})-B_\lambda(H_{kq|p}-H_{kp|q})]+ \\ +\gamma^{\tau k}\gamma^r_\lambda[(F_{kq}F_{rp}-F_{kp}F_{rq})-(H_{kq}H_{rp}-H_{kp}H_{rq})-R_{rkqp}]- \\ -A^\tau[\gamma^k_\lambda(F_{kp|q}-F_{kq|p})+B_\lambda(F_{kp}H^k_q-F_{kq}H^k_p)]+ \\ +B^\tau[\gamma^k_\lambda(H_{kp|q}-H_{kq|p})-A_\lambda(H_{kp}F^k_q-H_{kq}F^k_p)]. \quad (34)$$

Contracting by means of γ^q_τ, and using (3) and (4), we get

$$P_{\lambda p} \equiv \gamma^q_\tau P^\tau_{\lambda qp} = -A_\lambda F^q_{p|q}+B_\lambda H^q_{p|q}+\gamma^r_\lambda(-F^q_pF_{rq}+H^q_pH_{rq}+R_{pr}). \quad (35)$$

Contracting again, by means of $\gamma^{\lambda p}$, we get

$$P \equiv \gamma^{\lambda p}P_{\lambda p} = -F^q_pF^p_q+H^q_pH^p_q+R. \quad (36)$$

We now form the tensor

$$U_{\lambda p} \equiv P_{\lambda p}-\tfrac{1}{4}\gamma_{\lambda p}(P+R), \quad (37)$$

and, for field equations, write

$$U_{\lambda p} = 0. \quad (38)$$

In the Einstein-Mayer theory it was necessary to include an extra set of field equations, E.M. (47). In the present theory no such need

arises, since it turns out that the equivalent of E.M. (47) is already contained within the field equations (38). The value of the tensor $U_{\lambda p}$ is easily found to be

$$U_{\lambda p} = -A_\lambda F^s_{p|s} + B_\lambda H^s_{p|s} + \\ + \gamma^\lambda_\zeta [(R_{pr} - \tfrac{1}{2} g_{pr} R) + (g^{st} F_{ps} F_{rt} + \tfrac{1}{4} g_{pr} F^s_i F^t_s) - \\ - (g^{st} H_{ps} H_{rt} + \tfrac{1}{4} g_{pr} H^s_i H^t_s)]. \quad (39)$$

Thus, on multiplying the field equations (38) by γ^λ_q, A^λ, B^λ, respectively, we break them up into the three groups

$$\gamma^\lambda_q U_{\lambda p} \equiv (R_{pq} - \tfrac{1}{2} g_{pq} R) + (g^{st} F_{ps} F_{qt} + \tfrac{1}{4} g_{pq} F^s_i F^t_s) - \\ - (g^{st} H_{ps} H_{qt} + \tfrac{1}{4} g_{pq} H^s_i H^t_s) = 0, \quad (40)$$

$$A^\lambda U_{\lambda p} \equiv -F^s_{p|s} = 0, \quad (41)$$

$$B^\lambda U_{\lambda p} \equiv -H^s_{p|s} = 0. \quad (42)$$

In view of (23), the two sets (41) and (42) are the complete set of Maxwell equations, containing, therefore, the equivalent of E.M. (47); and, as in G.K.K., the set (40) reduces to the form

$$(R_{pq} - \tfrac{1}{2} g_{pq} R) + 2(g^{st} F_{ps} F_{qt} + \tfrac{1}{4} g_{pq} F^s_i F^t_s) = 0, \quad (43)$$

which is the usual set of stress-energy-momentum field equations of the general theory of relativity.

Thus we have constructed a four-dimensional theory that includes the motion of test bodies having magnetic charge in addition to electric charge and inertial mass; our theory does not contradict the usual equations of the general theory of relativity, it provides a needed extension of the Einstein-Mayer theory, and it avoids the necessity of introducing extraneous field equations of the type of E.M. (47).

II

7. Special Coordinates and Transformations

Apart from the obvious similarities of the geometries of G.K.K. and this paper, there exist deeper relationships that I shall now discuss. The treatment will be similar to that used* in demonstrating the connexion between the projective theory and the Einstein-Mayer field theory.

In each V_6 there is one coordinate-system that has special simplicity, namely, that in which four of the coordinate axes coincide with the

* Schouten and Van Danzig, *Proc. Ak. Wetensch. Amsterdam*, 34 (1931), 1398; Veblen, 'Projektive Relativitätstheorie', *Ergebn. der math. Wiss.* (Berlin) (1933), Chap. 8; Hoffmann, *Phys. Rev.* 43 (1933), 615.

dx^m of the corresponding tangent space of space-time, while the other two axes lie respectively along A^λ and B^λ. Since transformations in the V_6's have no physical significance, no generality will be lost in taking these coordinate-systems as standard. In such a system

$$\gamma_\lambda^p = \delta_\lambda^p, \qquad A^\lambda = \delta_0^\lambda, \qquad B^\lambda = \delta_\infty^\lambda. \tag{44}$$

I shall show that quantities of this type, and obeying the same algebraic and differential rules, arise in G.K.K.

To the coordinates of G.K.K. we apply the non-holonomic transformation given symbolically by

$$x^m = \bar{x}^m, \qquad x^0 = \bar{x}^0 - \int \phi_m \, d\bar{x}^m, \qquad x^\infty = \bar{x}^\infty + \int \psi_m \, d\bar{x}^m. \tag{45}$$

Then, by G.K.K. (16) and the law of transformation of $\kappa_{\lambda\mu}$, we find that the new components of the metric take the simple form

$$\bar{\kappa}_{mn} = g_{mn}, \qquad \bar{\kappa}_{00} = 1, \qquad \bar{\kappa}_{\infty\infty} = -1, \tag{46}$$

all other components vanishing.

It also follows that the non-zero components of $\bar{\kappa}^{mn}$ are given by

$$\bar{\kappa}^{mn} = g^{mn}, \qquad \bar{\kappa}^{00} = 1, \qquad \bar{\kappa}^{\infty\infty} = -1. \tag{47}$$

The connexion $K_{\mu\nu}^\lambda$ transforms according to the law

$$\bar{K}_{\mu\nu}^\lambda = K_{\sigma\tau}^\rho \frac{\partial x^\sigma}{\partial \bar{x}^\mu} \frac{\partial x^\tau}{\partial \bar{x}^\nu} \frac{\partial \bar{x}^\lambda}{\partial x^\rho} + \frac{\partial^2 x^\rho}{\partial \bar{x}^\nu \partial \bar{x}^\mu} \frac{\partial \bar{x}^\lambda}{\partial x^\rho}, \tag{48}$$

and, since (45) is non-holonomic, this shows that its symmetry will not be preserved. Hence $\bar{K}_{\mu\nu}^\lambda$ will not be merely the Christoffel symbols of the $\bar{\kappa}_{\lambda\mu}$, though the covariant derivative of $\bar{\kappa}_{\lambda\mu}$ will still vanish identically. From G.K.K. (18), the transformation law (48), and the transformation (45), we find that the non-zero components of $\bar{K}_{\mu\nu}^\lambda$ are

$$\left.\begin{aligned} \bar{K}_{mn}^l = \begin{Bmatrix} l \\ mn \end{Bmatrix}, \qquad \bar{K}_{mn}^0 = -\phi_{mn}, \qquad \bar{K}_{mn}^\infty = \psi_{mn} \\ \bar{K}_{m0}^l = \bar{K}_{0m}^l = \phi_m^l, \qquad \bar{K}_{m\infty}^l = \bar{K}_{\infty m}^l = \psi_m^l \end{aligned}\right\}, \tag{49}$$

where $\begin{Bmatrix} l \\ mn \end{Bmatrix}$ are the Christoffel symbols of the g_{mn}.

8. Algebraic Relations

In the new reference system

$$\bar{\kappa}_{\lambda 0} = \delta_{\lambda 0}, \qquad \bar{\kappa}_{\lambda\infty} = -\delta_{\lambda\infty}. \tag{50}$$

So, if we keep $\bar{x}^0$ and $\bar{x}^\infty$ fixed, the quantities $\bar{\kappa}_{\lambda 0}$, $\bar{\kappa}_{\lambda\infty}$ will behave as

vectors lying along $\bar{x}^0$ and $\bar{x}^\infty$. We denote them respectively by A_λ and B_λ, and shall relate them to the A_λ and B_λ already introduced in this paper. By (50) we see that

$$\left.\begin{aligned} \bar{\kappa}^{\lambda\mu}\bar{\kappa}_{\lambda 0}\bar{\kappa}_{\mu 0} = 1, \quad \bar{\kappa}^{\lambda\mu}\bar{\kappa}_{\lambda\infty}\bar{\kappa}_{\mu\infty} = -1 \\ \bar{\kappa}^{\lambda\mu}\bar{\kappa}_{\lambda 0}\bar{\kappa}_{\mu\infty} = 0, \quad \bar{\kappa}^{\lambda}_{m}\bar{\kappa}_{\lambda 0} = 0, \quad \bar{\kappa}^{\lambda}_{m}\bar{\kappa}_{\lambda\infty} = 0 \end{aligned}\right\}, \tag{51}$$

i.e.

$$A_\lambda A^\lambda = 1, \quad B_\lambda B^\lambda = -1, \quad A_\lambda B^\lambda = 0, \quad \bar{\kappa}^{\lambda}_{m}A_\lambda = 0, \quad \bar{\kappa}^{\lambda}_{m}B_\lambda = 0. \tag{52}$$

These relations show that A_λ and B_λ are orthogonal unit vectors perpendicular to space-time. The relations correspond respectively to the equations (6), (7), (5), (3), (4) of this paper, with $\bar{\kappa}^{\lambda}_{m}$ playing the part of γ^{λ}_{m}. The identification of $\bar{\kappa}^{\lambda}_{m}$ with γ^{λ}_{m} is in keeping with (44), since $\bar{\kappa}^{\lambda}_{m} = \delta^{\lambda}_{m}$; and, from (46), (47), it follows that the rules for raising and lowering suffixes of γ^{λ}_{m} are exactly the same as those applying to $\bar{\kappa}^{\lambda}_{m}$.

Again, we have

$$\begin{aligned} \delta^\mu_\lambda = \bar{\kappa}^\nu_\lambda \bar{\kappa}^\mu_\nu &= \bar{\kappa}^n_\lambda \bar{\kappa}^\mu_n + \bar{\kappa}^0_\lambda \bar{\kappa}^\mu_0 + \bar{\kappa}^\infty_\lambda \bar{\kappa}^\mu_\infty \\ &= \bar{\kappa}^n_\lambda \bar{\kappa}^\mu_n + \bar{\kappa}^{00}\bar{\kappa}_{0\lambda}\bar{\kappa}^\mu_0 + \bar{\kappa}^{\infty\infty}\bar{\kappa}_{\infty\lambda}\bar{\kappa}^\mu_\infty \\ &= \bar{\kappa}^n_\lambda \bar{\kappa}^\mu_n + A_\lambda A^\mu - B_\lambda B^\mu, \end{aligned}$$

so that

$$\bar{\kappa}^n_\lambda \bar{\kappa}^\mu_n = \delta^\mu_\lambda - A_\lambda A^\mu + B_\lambda B^\mu, \tag{53}$$

which is the same as (10).

Thus the basic algebraic formulae of the first part of this paper have been duplicated in the present section.

9. Differential Relations

Since the covariant derivative of $\bar{\kappa}_{\lambda\mu}$ vanishes,

$$\frac{\partial \bar{\kappa}^\lambda_\mu}{\partial x^n} + \bar{K}^\lambda_{\sigma n}\bar{\kappa}^\sigma_\mu - \bar{K}^\sigma_{\mu n}\bar{\kappa}^\lambda_\sigma \equiv 0. \tag{54}$$

For $\mu = m$, this gives, by (49),

$$\left[\frac{\partial \bar{\kappa}^\lambda_m}{\partial x^n} + \bar{K}^\lambda_{\sigma n}\bar{\kappa}^\sigma_m - \left\{\begin{matrix} s \\ mn \end{matrix}\right\}\bar{\kappa}^\lambda_s\right] = -\phi_{mn}A^\lambda + \psi_{mn}B^\lambda, \tag{55}$$

which is the same* as (13).

* Except that the F_{mn} and H_{mn} were not yet the curls of vectors.

Similarly, from $\bar{\kappa}_{\lambda\mu/\nu} \equiv 0$, for $\nu = n$, $\mu = 0$, on the one hand, and for $\nu = n$, $\mu = \infty$, on the other, we obtain

$$\left[\frac{\partial A_\lambda}{\partial x^n} - \bar{K}^\sigma_{\lambda n} A_\sigma\right] = \phi^s_n \bar{\kappa}_{\lambda s} = \phi_{sn} \bar{\kappa}^s_\lambda, \tag{56}$$

$$\left[\frac{\partial B_\lambda}{\partial x^n} - \bar{K}^\sigma_{\lambda n} B_\sigma\right] = \psi^s_n \bar{\kappa}_{\lambda s} = \psi_{sn} \bar{\kappa}^s_\lambda, \tag{57}$$

which are the same* as (18) and (19).

Thus both the algebraic and the differential fundamental relations obeyed by the A^λ, B^λ, $\bar{\kappa}^\lambda_m$ of the second part of this paper are the same as those obeyed by the A^λ, B^λ, γ^λ_m of the first part. The structures of the two geometries are, then, identical. But the type of transformation used to preserve this structure is different. In G.K.K. it is preserved by keeping x^0 and x^∞ unchanged, while in the generalized Einstein-Mayer theory it is preserved under quite general linear transformations by actually having the A_λ and B_λ transform as vectors.†

The field equations of the two theories differ in the numerical multipliers of the electromagnetic energy tensor, but this implies merely a difference in the physical units. Apart from this, and the difference in form, the field equations are physically equivalent.

10. Significance of the Field Equations (41) and (42)

As explained in G.K.K. § 4, there is a slight redundance in the scheme of G.K.K. in that ϕ_{mn} and ψ_{mn} are initially defined as curls, while this can also be deduced from the field equations G.K.K. (37), (38), and the condition G.K.K. (7) with ϕ_{mn} and ψ_{mn} not yet assumed to be curls. Such redundance is absent from the generalized Einstein-Mayer theory. In this theory the quantities $\Gamma^\lambda_{\mu 0}$, $\Gamma^\lambda_{\mu\infty}$ are not defined. If we define them so that

$$\Gamma^\lambda_{m0} = \Gamma^\lambda_{0m}, \quad \Gamma^\lambda_{m\infty} = \Gamma^\lambda_{\infty m}, \quad \Gamma^\lambda_{00} = \Gamma^\lambda_{0\infty} = \Gamma^\lambda_{\infty 0} = \Gamma^\lambda_{\infty\infty} = 0,$$

we obtain values for $\Gamma^\lambda_{\mu\nu}$ that are the same as those of $K^\lambda_{\mu\nu}$, except that F_{mn} and H_{mn} are not yet curls. But once it is decided from (41), (42), (23) that they are curls, it follows‡ that there exists a non-holonomic transformation that reduces $\Gamma^\lambda_{\mu\nu}$ to a symmetrical form that is identical with that of G.K.K.

* Except that the F_{mn} and H_{mn} were not yet the curls of vectors.

† Compare the definition of five-dimensional accompanying spaces given by Veblen, l.c. 56.

‡ Compare Hoffmann, *Phys. Rev.* 43 (1933), 615, § 6. The methods are so similar that we omit all details here.

ON TARRY'S PROBLEM (II)

By E. MAITLAND WRIGHT (*Aberdeen*)

[Received 2 August 1935]

THE notation used here is that of my former paper* with the same title; theorems and lemmas are numbered consecutively to those of that paper. We shall prove

THEOREM 8. $$M(k) \leqslant \frac{7k^2(k-11)^2}{72} + 56.$$

This is an improvement of Theorem 3. The method requires a knowledge of a good upper bound for $M(k)$ for certain small values of k; for this purpose we use the method of another paper.†

We take j any positive integer and write

$$[a_1,\ldots,a_j]_k = [b_1,\ldots,b_j]_k \tag{1}$$

to denote that

$$a_1^h+\ldots+a_j^h = b_1^h+\ldots+b_j^h \qquad (1 \leqslant h \leqslant k)$$
$$a_1^{k+1}+\ldots+a_j^{k+1} \neq b_1^{k+1}+\ldots+b_j^{k+1}.$$

If $j \geqslant M(k)$, there is a solution of (1), and, conversely, if there is a solution of (1), then $j \geqslant M(k)$.

We define the *generating polynomial* of any solution of (1) as

$$\phi_k(\theta) = \sum_{r=1}^{j} (\theta^{a_r} - \theta^{b_r}).$$

We write $$\phi_k^{(h)}(\theta) = \left(\theta\frac{d}{d\theta}\right)^h \phi_k(\theta).$$

Then it follows from (1) that $\phi_k^{(h)}(1) = 0$ if $1 \leqslant h \leqslant k$, but that $\phi_k^{(k+1)}(1) \neq 0$. Hence

$$\phi_k(\theta) = (\theta-1)^{k+1}\psi_k(\theta), \qquad \psi_k(1) \neq 0,$$

where $\psi_k(\theta)$ is a polynomial with integral coefficients.

If $j_1 = M(k_1), j_2 = M(k_2)$ and if $\phi_{k_1}(\theta), \phi_{k_2}(\theta)$ are the two generating polynomials of solutions of (1) for $k = k_1, j = j_1$, and $k = k_2, j = j_2$ respectively, it is clear that

$$\phi_{k_1}(\theta)\phi_{k_2}(\theta) = (\theta-1)^{k_1+k_2+2}\psi_{k_1}(\theta)\psi_{k_2}(\theta)$$

* *Quart. J. of Math.* (Oxford), 6 (1935) 261–7.

† *J. of London Math. Soc.* 9 (1934), 267–72.

is the generating polynomial of a solution of (1) with $k = k_1+k_2+1$, $j \leqslant 2j_1j_2$. Hence we have

LEMMA 7. $\quad M(k_1+k_2+1) \leqslant 2M(k_1)M(k_2).$

Tarry has observed that (1) implies that

$$[a_1,\ldots,a_j,\ b_1+d,\ldots,b_j+d]_{k+1} = [a_1+d,\ldots,a_j+d,\ b_1,\ldots,b_j]_{k+1}. \qquad (2)$$

This corresponds to the multiplication of $\phi_k(\theta)$ by $1-\theta^d$. We have at once

LEMMA 8. $\quad M(k+1) \leqslant 2M(k).$

If a number occurs on both sides of the equation in (2) it may be struck out. Applying Tarry's observation to the trivial

$$[1, 4]_1 = [2, 3]_1$$

with $d = 3,\ 5,\ 7,\ 4$ in succession and striking out equal pairs of numbers at each stage, we find solutions of (1) for $k = 2,\ 3,\ 4,\ 5$ with $j = k+1$. Dr. Eric Phillips has found a solution for $k = 6$ with $j = 7$, namely

$$[1,\ 19,\ 28,\ 59,\ 65,\ 90,\ 102]_6 = [2,\ 14,\ 39,\ 45,\ 76,\ 85,\ 103]_6$$

The result

$$[1,\ 5,\ 10,\ 24,\ 28,\ 42,\ 47,\ 51]_7 = [2,\ 3,\ 12,\ 21,\ 31,\ 40,\ 49,\ 50]_7$$

is due to Tarry. Applying Tarry's observation to this with $d = 9,\ 19,\ 17,\ 6$ in succession we find solutions for $k = 8,\ 9,\ 10,\ 11$ with $j = 10,\ 14,\ 14,\ 18$ respectively. Hence we have

LEMMA 9.

$$M(k) \leqslant 14 \quad (1 \leqslant k \leqslant 10), \qquad M(11) \leqslant 18.$$

LEMMA 10. *If* $r \geqslant 1$, $t \geqslant 1$, $rt \geqslant k$ *and* $j \geqslant N(rk)$, *then*

$$\{1, 2,\ldots, k\} \mapsto \{k+1, k+2,\ldots, k+t\}.$$

If

$$\{1, 2,\ldots, k\} \to \{k+1,\ldots, k+t),$$

then by Lemmas 3 and 4

$$\{1, 2,\ldots, rk\} \to \{1, 2,\ldots, r(k+t)\}.$$

Since $r(k+t) \geqslant (r+1)k$,

$$\{1, 2,\ldots, rk\} \to \{1, 2,\ldots, (r+1)k\}.$$

Hence, using Lemmas 3 and 4 again, we have

$$\{1, 2,\ldots, rk\} \to \{1, 2,\ldots, (r+1)(k+t)\}.$$

Since $(r+1)(k+t) > (r+2)k$, the process may be continued. Eventually we obtain

$$\{1, 2, \ldots, rk\} \to \{1, 2, \ldots, j\}.$$

Hence, by Lemma 2, $j < N(rk)$.

LEMMA 11. *If $r \geqslant 1$, $t \geqslant 1$ and $rt \geqslant k$, then for some s we have*

$$1 \leqslant s \leqslant t, \qquad M(k+s-1) \leqslant N(rk).$$

This is an immediate consequence of Lemma 10 and the definition of $M(k)$.

THEOREM 9. *If $r \geqslant 1$, $t \geqslant 1$ and $rt \geqslant k$, then*

$$M(k+t-1) \leqslant 2N(rk) \max_{1 \leqslant l \leqslant t-2} M(l).$$

If $s = t$ in Lemma 11, we have $M(k+t-1) \leqslant N(rk)$. If $s = t-1$, by Lemma 8,

$$M(k+t-1) \leqslant 2M(k+s-1) \leqslant 2N(rk).$$

If $1 \leqslant s \leqslant t-2$, by Lemma 7,

$$\begin{aligned} M(k+t-1) &\leqslant 2M(k+s-1)M(t-s-1) \\ &\leqslant 2N(rk) \max_{1 \leqslant l \leqslant t-2} M(l). \end{aligned}$$

We now put $t = 12$. By Lemma 9 and Theorems 1 and 9,

$$M(k+11) \leqslant 28N(rk) \leqslant 14\{(rk)^2+4\}.$$

We can always choose r so that

$$k \leqslant 12r \leqslant k+11.$$

Hence we have

$$M(k+11) \leqslant 14\left\{\frac{k^2(k+11)^2}{144}+4\right\}.$$

Changing our notation, we have for $k > 11$,

$$M(k) \leqslant 14\left\{\frac{k^2(k-11)^2}{144}+4\right\}.$$

The same is true for $k \leqslant 11$ by Lemma 9.

ON CERTAIN INTEGRAL FUNCTIONS OF ORDER ONE

By MARY L. CARTWRIGHT (*Cambridge*)

[Received 11 September 1935]

1. Introduction

In a recent paper Miss Young* has considered integral functions of the form

$$F(z) = \int_{a-0}^{b+0} e^{zt}\, dg(t), \tag{1.1}$$

where the integral is a Laplace-Stieltjes integral, and $g(t)$ is a complex-valued function of the real variable t with bounded variation in (a, b). Among other results Miss Young† proved

THEOREM 1. *If $F(z)$ is defined by* (1.1), *and if*

$$\lim_{n\to\infty} F(z_0+in\delta) = l \qquad (n = 1, 2,\ldots), \tag{1.2}$$

where δ is real, and $|\delta| < 2\pi/(b-a)$, then

$$\lim_{y\to\infty} F(z_0+iy\delta) = l. \tag{1.3}$$

The starting-point of this paper is a general theorem on integral functions of which this result is a special case. I shall consider integral functions of order 1 and type $k < \pi$ for which $|f(\pm n)|$ $(n = 1, 2,\ldots)$ is bounded, and functions which behave in a similar manner in the angle $|\arg z| \leqslant \alpha < \pi$. The series

$$\sum_{1}^{N} (-1)^n \frac{f(n)}{n^s}$$

will also be discussed.

I am indebted to Prof. J. M. Whittaker for suggestions which have enabled me to simplify the proofs of Theorems 2 and 3.

2. Integral functions of order 1 and type $k < \pi$

A simple transformation allows us to assume in (1.2) that $z_0 = 0$, $\delta > 0$. Further we may suppose that $b = -a = k$. For

$$F(z) = e^{\frac{1}{2}(a+b)z} \int_{-k}^{k} e^{\frac{1}{2}zt}\, dg_1(t) = e^{\frac{1}{2}(a+b)z} F_1(z),$$

* R. C. Young, *Math. Zeits.* 40 (1935), 292–311.

† Loc. cit. 305, Theorem 3 b.

where $\zeta = \frac{1}{2}(b-a)z/k$, $g_1(t) = g(\frac{1}{2}(a+b)+\frac{1}{2}(b-a)t/k)$; and there is no real loss of generality in considering $F_1(z)$ in place of $F(z)$. It is easy to show that

$$F_1(re^{i\theta}) = O(e^{kr|\cos\theta|}).$$

Now, writing iz in place of z, we see that (1.3) is a particular case of Theorem 3 below.

THEOREM 2. *Suppose that $f(z)$ is an integral function, such that*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} = k < \pi, \tag{2.1}$$

and

$$|f(\pm n)| < A \quad (n = 1, 2,\ldots). \tag{2.2}$$

Then

$$|f(x)| < KA \quad (K = K(k)) \tag{2.3}$$

for all real x.

THEOREM 3. *If $f(z)$ satisfies the hypotheses of Theorem 2, and also*

$$\lim_{n\to\infty} f(n) = l \quad (n = 1, 2,\ldots) \tag{2.4}$$

then

$$\lim_{x\to\infty} f(x) = l \quad (x > 0). \tag{2.5}$$

It is well known* that (2.1) and (2.3) imply

$$f(re^{i\theta}) = O(e^{kr|\sin\theta|}) \quad (0 \leqslant k < \pi), \tag{2.6}$$

uniformly for all θ.

3. The interpolation function

These two theorems are deduced from Tschakaloff's interpolation formula,† viz.

THEOREM 4. *If $f(z)$ satisfies the hypotheses of Theorem 2, then*

$$f(z) = \frac{\sin\pi z}{\pi}\left[\frac{f(0)}{z} + f'(0) + \sum_{n=-\infty}^{\infty}{}' (-1)^n f(n)\left\{\frac{1}{z-n} + \frac{1}{n}\right\}\right] \tag{3.1}$$

where $\sum'$ denotes that the term $n = 0$ is to be omitted.

The series (3.1) is absolutely convergent, and is similar to the cardinal function of interpolation.‡ In fact under the conditions stated

$$f(z) = \frac{\sin\pi z}{\pi}\left\{\frac{f(0)}{z} + \sum_{n=-\infty}^{\infty}{}' (-1)^n \frac{f(n)}{z-n}\right\}, \tag{3.2}$$

* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), 147, No. 325.

† L. Tschakaloff, *Jahresb. der Deutschen Math. Ver.* 43 (1934) 12, Zweite Lösung der Aufgabe 105. Tschakaloff proved (3.1) under a more stringent condition than (2.1), but his proof can easily be adapted to the more general case.

‡ See J. M. Whittaker, *Proc. Edinburgh Math. Soc.* (2) 1 (1928), 41–6, Theorems 1 and 3.

which is the cardinal function. This follows from

THEOREM 5. *If $f(z)$ satisfies the hypotheses of Theorem 2 then*

$$f'(0)+\sum_{n=-\infty}^{\infty}{}' (-1)^n\frac{f(n)}{n}=0. \tag{3.3}$$

This is easily proved by considering*

$$\int \frac{f(z)}{z\sin\pi z}\,dz$$

taken round the circle $|z|=N+\frac{1}{2}$, where N is an integer. Since (2.6) holds, the integral round the circle tends to 0 as $N\to\infty$, and the series on the left of (3.3) is the sum of the residues.

4. Proof of Theorem 2

Let p be an integer greater than $\pi/(\pi-k)$. Then $f(z)\sin(\pi z/p)$ satisfies the hypotheses of Theorem 4, and we have

$$f(z)=\frac{\sin\pi z}{\pi\sin(\pi z/p)}\left[f'(0)+\sum_{n=-\infty}^{\infty}{}' (-1)^n f(n)\sin\frac{\pi z}{p}\left\{\frac{1}{z-n}+\frac{1}{n}\right\}\right].$$

Let m be an integer, and write

$$s(z,p)=\frac{\sin\pi z}{\pi\sin(\pi z/p)}.$$

Then

$$\begin{aligned}\frac{f(z)}{s(z,p)}-\frac{f(mp)}{s(mp,p)}&=\left\{\sum_{-\infty}^{(m-1)p}{}'+\sum_{(m-1)p+1}^{mp-1}+\sum_{mp+1}^{(m+1)p}+\sum_{(m+1)p+1}^{\infty}\right\}\frac{(-1)^n(mp-z)f(n)}{(z-n)(mp-n)}\sin\frac{\pi z}{p}\\&=S_1+S_2+S_3+S_4.\end{aligned}$$

If $z=x+iy$, $y=\pm1$, $(m-\frac{1}{2})p\leqslant x\leqslant(m+\frac{1}{2})p$, we have in S_1

$$\left|\frac{mp-z}{(z-n)(mp-n)}\right|\leqslant\frac{(1+p^2)^{\frac{1}{2}}}{\{(m-\frac{1}{2})p-n\}(mp-n)};$$

and if $-1\leqslant y\leqslant1$, $x=(m\pm\frac{1}{2})p$, the same result holds. Hence on the boundary of the rectangle $-1\leqslant y\leqslant1$, $(m-\frac{1}{2})p\leqslant x\leqslant(m+\frac{1}{2})p$, we have

$$|S_1|\leqslant 2Ap\sum_{1}^{\infty}\frac{1}{n^2}=\tfrac{1}{3}\pi^2Ap.$$

In S_2 and S_3 we have, for $n\neq mp$,

$$\left|\frac{mp-z}{(z-n)(mp-n)}\right|\leqslant(1+p^2)^{\frac{1}{2}}<2p.$$

* Compare § 9.

on the boundary of the rectangle; and in S_4

$$\left|\frac{mp-z}{(z-n)(mp-n)}\right| \leqslant \frac{(1+p^2)^{\frac{1}{2}}}{\{n-(m+\frac{1}{2})p\}(n-mp)}.$$

Hence on the boundary of the rectangle

$$|S_2| \leqslant 2p^2A, \qquad |S_3| \leqslant 2p^2A,$$

$$|S_4| \leqslant 2Ap \sum_1^\infty \frac{1}{n^2} = \tfrac{1}{3}\pi^2 Ap;$$

and so, on the boundary of the rectangle,

$$\left|\frac{f(z)}{s(z,p)}\right| \leqslant \left|\frac{f(mp)}{s(mp,p)}\right| + |S_1| + |S_2| + |S_3| + |S_4|$$
$$\leqslant K_1 A,$$

where K_1 depends on p but not on m. It is easy to see that for $-1 \leqslant y \leqslant 1$ we have

$$|s(z,p)| < K_2 = K_2(p);$$

and so

$$|f(z)| < K_3 A = K_3(p)A$$

on the boundary of the rectangle. Since $f(z)$ is regular inside the rectangle, it follows that the result holds inside the rectangle; and, since p depends only on k, we have the required result.

5. Proof of Theorem 3

We suppose, as we obviously may, that $l = 0$. Then, given any positive ϵ_1, we can choose N_0 so that $|f(n)| < \epsilon_1$, for $n > N_0$, and the results of the last paragraph for S_2, S_3, S_4 give

$$|S_2| + |S_3| + |S_4| \leqslant \epsilon_1 K_4 p^2,$$

where K_4 is independent of z, m, p, on the boundary of the rectangles $-1 \leqslant y \leqslant 1$, $(m-\frac{1}{2})p \leqslant x \leqslant (m+\frac{1}{2})p$, $(m-1)p > N_0$. We also have

$$|S_1| \leqslant (1+p^2)^{\frac{1}{2}} \left\{\sum_{-\infty}^{N_0} + \sum_{N_0+1}^{(m-1)p}\right\} \frac{|f(n)|}{\{(m-\frac{1}{2})p-n\}(mp-n)}$$

$$< 2Ap \sum_{(m-1)p-N_0}^{\infty} \frac{1}{n^2} + 2\epsilon_1 p \sum_1^{(m-1)p-N_0} \frac{1}{n^2} < \epsilon_2$$

on the boundary of the rectangle for every positive ϵ_2, provided that ϵ_1 is sufficiently small and $(m-1)p-N_0$ sufficiently large. Hence

$$\left|\frac{f(z)}{s(z,p)}\right| \leqslant \left|\frac{f(mp)}{s(mp,p)}\right| + \epsilon_1 K_4 p^2 + \epsilon_2;$$

and, since $|f(mp)| \to 0$ as $m \to \infty$, we have

$$|f(z)/s(z,p)| < \epsilon$$

on the boundary of the rectangle

$$-1 \leqslant y \leqslant 1, \qquad (m-\tfrac{1}{2})p \leqslant x \leqslant (m+\tfrac{1}{2})p$$

for every positive ϵ, provided that $m > m_0(\epsilon, p)$. It follows as before that

$$|f(z)| < \epsilon K_5$$

where K_5 depends on p, for $-1 \leqslant y \leqslant 1$, $(m-\frac{1}{2})p \leqslant x \leqslant (m+\frac{1}{2})p$, provided that $m > m_0(\epsilon, p)$; and this includes the required result.

6. Functions regular in an angle

These results can all be extended to functions regular in an angle less than π. We cannot of course obtain an exact expression such as (3.1) for the function, but we obtain an approximate expression which answers our purpose very nearly as well.

THEOREM 6. *Suppose that*

(i) $f(z)$ *is regular for* $|\arg z| \leqslant \alpha < \pi$;

(ii) $|f(n)| < A \quad (n = 1, 2, \ldots)$;

(iii) $|f(re^{i\theta})| < Be^{\pi r \tan\alpha \cos\theta}$ *for* $|\theta| \leqslant \alpha$.

Then for $|\theta| \leqslant \alpha$, *we have*

$$|f(re^{i\theta}) - \phi(re^{i\theta})| < K_6 \log r \, e^{\pi r \sin\theta},$$

where

$$\phi(z) = \frac{\sin \pi z}{\pi} \sum_{n=1}^{\infty} (-1)^n f(n) \left(\frac{1}{z-n} + \frac{1}{n}\right), \tag{6.1}$$

and K_6 *depends on* A, B, α *only*.

Condition (iii) may be replaced by

$$|f(re^{\pm i\alpha})| < Be^{\pi r \sin\alpha}$$

and $|f(re^{i\theta})| < e^{\epsilon r \frac{1}{2}\pi/\alpha}$ $(|\theta| \leqslant \alpha)$ for every positive ϵ and every r greater than $r_0(\epsilon)$. For by a theorem of Phragmèn and Lindelöf* these two conditions imply (iii).

THEOREM 7. *If* $f(z)$ *satisfies the hypotheses of Theorem* 6,

$$|f(re^{i\theta})| < K_7 \log r \, e^{\pi r \sin\theta},$$

where $K_7 = K_7(A, B, \alpha)$, *for* $|\theta| \leqslant \alpha$.

THEOREM 8. *If* $f(z)$ *satisfies the hypotheses of Theorem* 6, *and if*

$$\left| \sum_{n=1}^{N} (-1)^n \frac{f(n)}{n} \right| < C, \tag{6.2}$$

then

$$|f(re^{i\theta}) - \phi(re^{i\theta})| < K_8 e^{\pi r \sin\theta},$$

where $K_8 = K_8(A, B, C, \alpha)$, *for* $|\theta| \leqslant \alpha$.

* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), 325.

THEOREM 9. *If* (i) *and* (ii) *of Theorem* 6 *hold, and*

$$|f(re^{i\theta})| < Be^{kr\tan\alpha\cos\theta}, \tag{iii)'}$$

where $k < \pi$, *for* $|\theta| \leqslant \alpha$, *then*

$$\sum_{n=1}^{\infty} (-1)^n \frac{f(n)}{n}$$

is convergent.

THEOREM 10. *If the hypotheses of Theorem* 9 *hold, then*

$$|f(x)| < K_9 = K_9(A, B, \alpha, k)$$

for $x > 0$.

Theorems 6, 7, 8, 9 are essential to the proof of 10, and it is evident that Theorem 2 with the constant slightly modified can be deduced from Theorem 10. These theorems have refinements corresponding to Theorem 3, and similar theorems can be obtained if we replace the hypothesis (ii) by

$$|f(a_n)| < A\psi(n),$$

where $0 < |a_1| < |a_2| < \ldots < |a_n| \to \infty$, $|\arg a_n| < \alpha$, and $\psi(n)$ is a positive monotonic function tending to 0 or ∞. Rather stringent conditions must be laid on the sequence a_n as regards the regularity of distribution as well as the thickness of distribution in order to obtain significant results. A theorem of this nature has been proved by V. Bernstein.*

Putting $z = w^\rho$ in Theorems 6–10 we obtain results for a function $F(w)$ which is of order ρ in the angle $|\arg w| \leqslant \alpha/\rho < \pi/\rho$, and the hypothesis (ii) becomes $|F(n^{1/\rho})| < A$. J. M. Whittaker† has proved a theorem, similar in some respects to Theorem 10, for functions of order 2 which are bounded at the lattice points $w = m+ni$, m and n being integers; and it has been shown‡ that integral functions of order 2 and minimum type which are bounded at the lattice points reduce to a constant.

A very general theorem of this kind for functions regular in the unit circle has been proved by J. M. Whittaker.§

* V. Bernstein, *Leçons sur les séries de Dirichlet* (Paris, 1933), 230.

† J. M. Whittaker, *Proc. London Math. Soc.* (2) 37 (1934), 383–401, Theorem 4.

‡ See J. M. Whittaker, loc. cit. 397–9, and G. Pólya, *Jahresb. der Deutschen Math. Ver.* 43 (1933), 67–9. Bemerkungen zu der Lösung der Aufgabe 105.

§ See J. M. Whittaker, *Interpolatory Function Theory* (*Cambridge Math. Tracts* No. 33), 57.

7. Properties of the interpolation function

We require certain lemmas about the function $\phi(z)$.

LEMMA 1. *If $\phi(z)$ is defined by* (6.1), *and if* (ii) *of Theorem* 6 *holds, then $\phi(n) = f(n)$, and*

$$|\phi(re^{i\theta})| \leqslant K_{10} A e^{\pi r \sin\theta} \log r \qquad (0 \leqslant |\theta| \leqslant \tfrac{1}{2}\pi). \tag{7.1}$$

If, in addition, (6.2) *holds, then*

$$|\phi(re^{i\theta})| \leqslant K_{11}(A+C)e^{\pi r \sin\theta} \qquad (0 < \delta \leqslant |\theta| \leqslant \tfrac{1}{2}\pi), \tag{7.2}$$

where $K_{11} = K_{11}(\delta)$,

Proof. It is easy to see that $\phi(n) = f(n)$. Let $z = re^{i\theta}$,

$$N-\tfrac{1}{2} \leqslant r \leqslant N+\tfrac{1}{2},$$

where N is an integer, and write

$$\begin{aligned}\sum_1^\infty (-1)^n f(n)\left\{\frac{1}{z-n}+\frac{1}{n}\right\} &= \sum_1^{2N} (-1)^n \frac{f(n)}{z-n} + \sum_1^{2N} (-1)^n \frac{f(n)}{n} + \sum_{2N+1}^{\infty} (-1)^n \frac{f(n)z}{n(z-n)} \\ &= S_1+S_2+S_3. \end{aligned} \tag{7.3}$$

In S_3, $|z-n| \geqslant n-(N+\frac{1}{2})\cos\theta \geqslant n-(N+\frac{1}{2})$, and so

$$|S_3| \leqslant 2A\,N \sum_{2N+1}^{\infty} \frac{1}{n(n-N-\frac{1}{2})} \leqslant 4A\,N \sum_{2N+1}^{\infty} \frac{1}{n^2} < K_{12} A. \tag{7.4}$$

Also
$$|S_2| \leqslant 2A \log N. \tag{7.5}$$

If $0 < \delta \leqslant |\theta| \leqslant \frac{1}{2}\pi$, then $|z-n| \geqslant (N-\frac{1}{2})\sin\delta$, and so

$$|S_1| \leqslant \frac{2AN}{(N-\frac{1}{2})\sin\delta} < \frac{3A}{\sin\delta}. \tag{7.6}$$

Combining (7.4), (7.5), (7.6) with (7.3), we have (7.1) for $\delta \leqslant |\theta| \leqslant \frac{1}{2}\pi$ with K_{10} which depends on δ. Also, using (6.2) in place of (7.5), we have (7.2).

Taking $\delta = \frac{1}{4}\pi$, we have (7.1) with an absolute constant K_{10} for $\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Now suppose that $0 \leqslant |\theta| \leqslant \frac{1}{4}\pi$, and let

$$N_1-\tfrac{1}{2} \leqslant r\cos\theta \leqslant N_1+\tfrac{1}{2},$$

where N_1 is an integer. Then

$$N < r+\tfrac{1}{2} < 2r\cos\tfrac{1}{4}\pi < 2N_1+1$$

and

$$\begin{aligned} |z-n| &\geqslant N_1-n \qquad (n = 1, 2, \ldots, N_1-1), \\ |z-n| &\geqslant n-N_1 \qquad (n = N_1+1, N_1+2, \ldots). \end{aligned}$$

Hence

$$\begin{aligned}|S_1| &\leqslant A\sum_1^{2N}\frac{1}{|z-n|}\\ &\leqslant A\left\{\sum_1^{N_1-1}\frac{1}{N_1-n}+\sum_{N_1+1}^{2N}\frac{1}{n-N_1}+\frac{1}{|z-N_1|}\right\}\\ &\leqslant 4A\log N_1+A/|z-N_1|\\ &\leqslant 4A\log N+\tfrac{1}{2}A \qquad (7.7)\end{aligned}$$

except in the square $N_1-\frac{1}{2}\leqslant x\leqslant N_1+\frac{1}{2}$, $-\frac{1}{2}\leqslant y\leqslant\frac{1}{2}$ $(z=x+iy)$. It follows that (7.1) holds for $0\leqslant|\theta|\leqslant\frac{1}{2}\pi$, except in the squares $n-\frac{1}{2}\leqslant x\leqslant n+\frac{1}{2}$, $-\frac{1}{2}\leqslant y\leqslant\frac{1}{2}$ $(n=1,2,\ldots)$. Since $\phi(z)$ is regular in these squares, the complete result follows from the maximum-modulus principle.

8. Proofs of Theorems 6, 7, and 8

Consider the function

$$\Phi(z)=\frac{f(z)-\phi(z)}{\log z\sin\pi z}.$$

Since $\phi(n)=f(n)$, $\Phi(z)$ is regular for $|\arg z|\leqslant\alpha$, $|z|\geqslant 1$; and by the lemma and condition (iii) we have

$$\begin{aligned}|\Phi(re^{\pm i\alpha})| &< K_{11}=K_{11}(A,B,\alpha) \qquad (r\geqslant 1),\\ |\Phi(re^{i\theta})| &= K_{12}e^{\pi r\tan\alpha}\end{aligned}$$

for $|\theta|\leqslant\alpha$, $r=n+\frac{1}{2}$ $(n=1,2,\ldots)$. Hence by a theorem due to Phragmèn and Lindelöf

$$|\Phi(re^{i\theta})|<K_{13} \quad (|\theta|\leqslant\alpha;\, r\geqslant 1). \qquad (8.1)$$

and Theorem 6 follows at once. Theorem 7 follows from Theorem 6 and (7.1).

Theorem 8 follows from (7.2) in a similar manner if we consider

$$\Phi_1(z)=\{f(z)-\phi(z)\}\operatorname{cosec}\pi z$$

in place of $\Phi(z)$.

9. Proof of Theorem 9

It follows from Theorem 7 that

$$|f(x)|<K_7\log x \qquad (x>0). \qquad (9.1)$$

and by (iii)′ we have

$$|f(re^{i\theta})|<Be^{kr\tan\alpha\cos\theta} \qquad (|\theta|\leqslant\alpha). \qquad (9.2)$$

Hence, applying the theorem of Phragmèn and Lindelöf to

$$f(z)e^{\pm kiz}/\log z$$

in the angles $\quad -\alpha \leqslant \arg z \leqslant 0,\ 0 \leqslant \arg z \leqslant \alpha,$

we have $\quad |f(re^{i\theta})| < K_{14}\log r e^{kr\sin\theta} \qquad (|\theta| \leqslant \alpha).$

Now consider the integral

$$I_N = \frac{1}{\epsilon\pi i}\int \frac{f(z)\,dz}{z\sin\pi z}$$

taken round the sector $|\arg z| \leqslant \alpha,\ \frac{1}{2} \leqslant |z| \leqslant N+\frac{1}{2}$, where N is an integer. By the residue theorem

$$I_N = \sum_1^N (-1)^n \frac{f(n)}{n}, \tag{9.3}$$

and the integral taken along $\arg z = \pm\alpha$ converges as $N \to \infty$ in virtue of (9.2). Also

$$\left|\int_{-\alpha}^{\alpha} \frac{f\{(N+\frac{1}{2})e^{i\theta}\}\,d\theta}{\sin\{\pi(N+\frac{1}{2})e^{i\theta}\}}\right| \leqslant K_{15}\log(N+\tfrac{1}{2})\int_0^{\alpha} e^{(k-\pi)(N+\frac{1}{2})\sin\theta}\,d\theta$$

$$< K_{15}\log(N+\tfrac{1}{2})\int_0^{\alpha} e^{(k-\pi)\frac{2}{\pi}(N+\frac{1}{2})\theta}\,d\theta$$

$$\leqslant \frac{K_{15}\,\pi\log(N+\frac{1}{2})}{2(\pi-k)(N+\frac{1}{2})} \to 0$$

as $N \to \infty$. Hence the series in (9.3) converges as $N \to \infty$.

10. Proof of Theorem 10

Choose an integer p greater than $\pi/(\pi-k)$ and apply Theorem 9 to $f(z)\sin(\pi z/p)$. We find that

$$\sum_1^{\infty} (-1)^n \frac{f(n)}{n}\sin\frac{\pi n}{p}$$

is convergent; and so, by Theorem 8,

$$|f(re^{i\theta}) - \phi_p(re^{i\theta})| < K_8\, e^{\pi r\sin\theta}\,|\mathrm{cosec}(\pi re^{i\theta}/p)|,$$

where $\quad \phi_p(re^{i\theta}) = \dfrac{\sin\pi z}{\pi\sin(\pi z/p)}\displaystyle\sum_1^{\infty}(-1)^n f(n)\sin\frac{\pi n}{p}\left\{\frac{1}{z-n}+\frac{1}{n}\right\},$

for $|\theta| \leqslant \alpha$.

Let $z = x+iy$. Then on the boundary of the rectangle

$$(m-\tfrac{1}{2})p \leqslant x \leqslant (m+\tfrac{1}{2})p, \qquad -1 \leqslant y \leqslant 1,$$

where m is an integer, $\mathrm{cosec}(\pi z/p)$ is bounded, and so by the

maximum-modulus principle $|f(z)-\phi_p(z)|$ is bounded inside the rectangle. Since $|f(mp)|$ is bounded, $|\phi_p(mp)|$ is bounded as $m \to \infty$. In the rectangle

$$(m-\tfrac{1}{2})p \leqslant x \leqslant (m+\tfrac{1}{2})p, \qquad -1 \leqslant y \leqslant 1,$$

we have

$$|f(z)-\phi_p(z)-f(mp)-\phi_p(mp)| \leqslant |f(z)-\phi_p(z)|+|f(mp)-\phi_p(mp)| < K_{16}.$$

Hence

$$|f(z)| < K_{16}+|f(mp)|+|\phi_p(z)-\phi_p(mp)| < K_{17}+|\phi_p(z)-\phi_p(mp)|.$$

The proof of Theorem 2 shows that $|\phi_p(z)-\phi_p(mp)| < K_{18}$; and so we have the required result.

11. The series $\sum (-1)^n f(n) n^{-s}$

It may be observed that the method of § 9 applied to

$$\int \frac{f(z)\,dz}{z^s \sin \pi z}$$

shows that, under the hypotheses of Theorem 9,

$$\sum_1^N (-1)^n \frac{f(n)}{n^s}$$

is convergent for $s > 0$, and

$$\sum_1^N (-1)^n \frac{f(n)}{n^s} = O(N^{-s})$$

for $s \leqslant 0$.

THE NUMBER OF REPRESENTATIONS OF A NUMBER AS A SUM OF n NON-NEGATIVE nTH POWERS

By S. CHOWLA (*Waltair, India*) and
S. S. PILLAI (*Annamalainagar, India*)

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1. LET $r_{s,n}(N)$ denote the number of representations of N as a sum of s non-negative nth powers. Hardy and Littlewood conjectured that for fixed n ($\geqslant 3$),

$$r_{n,n}(N) = O(N^{\epsilon}) \text{ for any positive } \epsilon. \tag{1}$$

We shall prove the following result in the opposite direction.

THEOREM*. *For fixed n ($\geqslant 5$) we have*

$$r_{n,n}(N) = \Omega(\log\log N)$$

i.e. there is a positive number c_1, depending only on n, such that

$$r_{n,n}(N) > c_1 \log\log N,$$

is true for infinitely many N.

2. In the sequel we shall denote by $c_2, c_3, \ldots, c_{19}$, *positive constants depending only on n.*

Notation. Let ρ be a primitive qth root of unity,

$$S_\rho = \sum_{h=0}^{q-1} e^{2\pi i \frac{a}{q} h^n} = \sum_{h=0}^{q-1} \rho^{h^n} \quad [(a,q)=1], \qquad A(q) = q^{-s} \sum_{\rho} (S_\rho)^s e^{-2\pi i \frac{m}{q} N},$$

$$S(N;n;s) = \sum_{q=1}^{\infty} A(q) = \prod_p \chi_p \quad (s \geqslant 5),$$

where $\chi_p = 1 + A(p) + A(p^2) + \ldots$ and p runs through all primes; p_r denotes the rth prime. $S(N;n;s)$ is the singular series in Waring's problem. See Landau, *Vorlesungen über Zahlentheorie*, Band 1, 281–302, where the first four of the following lemmas are proved.

LEMMA 1. (i) $S_\rho = p^{l-1}$ $(q = p^l;\ p \nmid n;\ 2 \leqslant l \leqslant n)$;

(ii) $|S_\rho| \leqslant (n-1)\sqrt{p}$ $(q = p)$;

(iii) $S_\rho = p^{n-1} S_{\rho^{p^n}}$ $(q = p^l;\ l > n)$.

(*Clearly ρ^{p^n} is a primitive p^{l-n}th root of unity.*)

* For $n = 5$ Chowla has previously proved a sharper result. See the paper by Sastry in *J. of London Math. Soc.* 9 (1934), 242–6. For a weaker form of the theorem of this paper see *Proc. Indian Ac. Sc.* (A), 2 (1935), 397–401.

We shall use $p^\theta \| m$ to denote that $p^\theta | m$ but $p^{\theta+1} \nmid m$. Then we have

LEMMA 2. *If* (i) $p^{\beta n+\sigma} \| N$ *where* $\beta \geqslant 0, n > \sigma \geqslant 0$, *and* (ii) $p^\theta \| n$, *and* $\gamma = \theta+1$ $(p > 2)$, *or* $\gamma = \theta+2$ $(p = 2)$, *then* $A(p^l) = 0$ *for* $l > l_0$ *where*

$$l_0 = \max(\beta n+\sigma+1,\ \beta n+\gamma).$$

LEMMA 3. $1+A(p)+\ldots+A(p^l) = p^{-(s-1)l} M(p^l, N)$ *where* $M(p^l, N)$ *is the number of solutions of* $\sum_{m \leqslant s} h_m^n \equiv N \pmod{p^l}$ *and* $0 \leqslant h_m < p^l$.

LEMMA 4. *For* $s = n+1$ $(n \geqslant 4)$ *there exists a number* c_2, *such that*

$$\chi_p > 1-p^{-\frac{1}{2}} \quad \text{for } p \geqslant c_2.$$

3. LEMMA 5. *Let* $n \geqslant 5$ *and* $N \equiv p_r^n \pmod{p_r^{2n}}$ *so that* $p_r^n \| N$. *Then there exists a number* c_3 *such that, for* $s = n+1$, $r > c_3$,

$$\chi_{p_r} = 1+\frac{1}{p_r}+\frac{b}{p_r^{\frac{3}{2}}} \qquad (|b| < 1).$$

Proof. First of all we find c_4 so large that $(p_r, n) = 1$ for all r not less than c_4. For such r, since $p_r^n \| N$, Lemma 2 shows that

$$\chi_{p_r} = 1+A(p_r)+\ldots+A(p_r^{n+1}). \tag{2}$$

By Lemma 1 (i), for $s = n+1$,

$$A(p_r^m) = p_r^{m-n-1}-p_r^{m-n-2} \quad (2 \leqslant m \leqslant n). \tag{3}$$

By Lemma 1 (ii), for $s = n+1$,

$$|A(p_r)| \leqslant \frac{c_5 p_r^{\frac{1}{2}(n+1)} p_r}{p_r^{n+1}} < \frac{c_6}{p_r^2} \quad (n \geqslant 5). \tag{4}$$

By Lemma 1, (iii) and (ii), for $s = n+1$,

$$|A(p_r^{n+1})| \leqslant \frac{c_7 (p_r^{n-\frac{1}{2}})^{n+1} p_r^{n+1}}{p_r^{(n+1)(n+1)}} = \frac{c_7}{p_r^{\frac{1}{2}(n+1)}} < \frac{c_8}{p_r^2} \qquad (n \geqslant 3). \tag{5}$$

From (2), (3), (4), (5) we immediately obtain Lemma 5.

LEMMA 6. *For* $r \leqslant c_3$ *(of Lemma 5) let* $p_r^{\theta_r} \| n$. *Let* $Y = 2^{\theta_1+2} p_2^{\theta_2+1} \ldots p_m^{\theta_m+1}$ *where* $m = [c_3]$. *Then for all* $N \equiv 1 \pmod{Y}$, *and* $s = n+1$,

$$\prod_{r \leqslant c_3} \chi_{p_r} > c_9.$$

Proof. By Lemmas 2 and 3, if $s = n+1$, $N \equiv 1 \pmod{Y}$,

$$\chi_2 = 1+A(2)+\ldots+A(2^{\theta_1+2}) = 2^{-(\theta_1+2)n} M(2^{2+\theta_1}, N), \tag{6}$$

$$\begin{aligned} \chi_{p_r} &= 1+A(p_r)+\ldots+A(p_r^{\theta_r+1}) \\ &= 2^{-(\theta_r+1)n} M(p_r^{\theta_r+1}, N) \quad (2 \leqslant r \leqslant m). \end{aligned} \tag{7}$$

But, since $N \equiv 1 \pmod{2^{\theta_1+2} p_2^{\theta_2+1} \ldots p_m^{\theta_m+1}}$, we have

$$M(2^{\theta_1+2}, N) \geqslant 1, \qquad M(p_r^{\theta_r+1}, N) \geqslant 1 \quad (2 \leqslant r \leqslant m). \tag{8}$$

From (6), (7), (8) we obtain Lemma 6 since θ_r $(r \leqslant m)$ depends only on n.

LEMMA 7. $$\prod_{p \leqslant x} \left(1-\frac{1}{p}\right) \sim \frac{e^{-C}}{\log x},$$

where C is Euler's constant.

The proof is well known.

LEMMA 8. $$\prod_{p \leqslant x} \left(1+\frac{1}{p}\right) > c \log x,$$

where c is an absolute positive constant.

Proof. LEMMA 7.

Let $$z = c_{10} \log x, \tag{9}$$

where later we make $x \to \infty$ and c_{10} will be subjected to certain restrictions. It follows from Lemma 5 that, if $N \equiv p_r^n \pmod{p_r^{2n}}$ for $c_3 \leqslant r \leqslant z$, then

$$\prod_{c_3 < r \leqslant z} \chi_{p_r} > c_{11} \log(z \log z) > c_{12} \operatorname{loglog} x, \tag{10}$$

by Lemma 8. Now let us take $z > c_2$ (of Lemma 4). Then*

$$\begin{aligned} S(N; n; n+1) &= \prod_{r \leqslant c_3} \chi_{p_r} \prod_{c_3 < r \leqslant z} \chi_{p_r} \prod_{r > z} \chi_{p_r} \\ &> c_9(c_{12} \operatorname{loglog} x) c_{13} \quad \text{by Lemma 6, (10), and Lemma 4} \\ &> c_{14} \operatorname{loglog} x, \end{aligned} \tag{11}$$

where N is subject to the restrictions

$$N \equiv 1 \pmod{2^{\theta_1+2} p_2^{\theta_2+1} \ldots p_m^{\theta_m+1}} \tag{12}$$

where $m = [c_3]$ and the θ's depend only on n, and

$$N \equiv p_r^n \pmod{p_r^{2n}} \quad (c_3 < r \leqslant z). \tag{13}$$

Now, from (9), for $x > c_{19}$,

$$x^{nc_{10}} < 2^{\theta_1+2} \prod_{2 \leqslant r \leqslant m} \{p_r^{\theta_r+1}\} \prod_{c_3 \leqslant r \leqslant z} \{p_r^{2n}\} < x^{3nc_{10}}. \tag{14}$$

Hence, if $$3nc_{10} < 1, \tag{15}$$

the number of N's satisfying (12), (13) *and*

$$1 \leqslant N \leqslant x, \tag{16}$$

is greater than $c_{15} x^{1-4nc_{10}}$.

* Here c_{11}, c_{12}, c_{13}, c_{14} will depend on our choice of c_{10}; c_{10} itself is subject to (15) and (17).

Again, we have from Theorem 5 of P.N. VI of Hardy and Littlewood*

LEMMA 9. *There exists a positive number c_{16}, depending only on n, such that, if* (1) *is true, then*

$$r_{n+1,n}(N) = ES(N;n;n+1)N^{\frac{1}{n}} + O\left(N^{\frac{1}{n}}\right)$$

(where E is positive and depends only on n) is false for at most $x^{1-c_{16}}$ values of $N \leqslant x$.

Let us choose c_{10} in (9) so that (15) is true and

$$4nc_{10} < c_{16}. \tag{17}$$

Hence the number of solutions (in N) of (12), (13), (16) is greater than

$$c_{15}x^{1-c_{16}+\delta},$$

where, by (17), $\delta = c_{16} - 4nc_{10} > 0$. Since $x \geqslant N$ it now follows† from (11) and Lemma 9 that

LEMMA 10. *There exists a positive number c_{17}, depending only on n, such that, if* (1) *is true, then*

$$r_{n+1,n}(N) > c_{17}(\log\log N)N^{\frac{1}{n}},$$

is true for infinitely many N.

From Lemma 10 it follows immediately that

LEMMA 11. *If* (1) *is true, there exists a positive number c_{18}, depending only on n, such that*

$$r_{n,n}(N) > c_{18}\log\log N$$

is true for infinitely many N.

If possible, let $$r_{n,n}(N) = o(\log\log N), \tag{18}$$

then, obviously, (1) is true, and hence by Lemma 11,

$$r_{n,n}(N) > c_{18}\log\log N \text{ for infinitely many } N. \tag{19}$$

But (19) contradicts (18). Hence (18) is false and our theorem is proved.

* *Math. Zeits.* 23 (1925), 1–37.

† Since $c_{15}x^{1-c_{16}+\delta} > x^{1-c_{16}}$ for $x > x_0(n)$.

ON SKEW POLYGONS WHOSE SIDES ARE, ALTERNATELY, GENERATORS OF TWO QUADRIC SURFACES

By A. L. DIXON (*Oxford*)

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Two triads of points $ABC, A'B'C'$, each triad taken in a definite cyclic order, determine three skew hexagons $AC'BA'CB'$, $AA'BB'CC'$, $AB'BC'CA'$; three quadrics through the alternate sides of the hexagons; and three Schur quadrics with regard to which the opposite sides of the hexagons are polar lines.

The geometry of this figure has received some attention, chiefly because a skew hexagon is half of a Schläfli double six.

In this paper I make use of the elliptic coordinates of a point on a quadric surface to investigate the geometry of a skew hexagon and its associated quadrics, and also consider the more general case of a skew polygon of $4n+2$ sides whose alternate sides are generators of two quadrics of a pencil.

The principal results are given in § 6. In §§ 7–11 I consider the invariant relations between some of the associated quadrics, and find, explicitly, the invariant relation for a skew decagon which corresponds to $\Theta\Theta' = 4\Delta\Delta'$ for a skew hexagon. The addition-formula for the elliptic functions on which this section depends is curious.

1. The investigation begins with the following formula in elliptic functions:

$$\begin{aligned}&\operatorname{dn}A\operatorname{dn}B\operatorname{dn}C\operatorname{dn}D - k^2\operatorname{cn}A\operatorname{cn}B\operatorname{cn}C\operatorname{cn}D + \\ &\qquad + k^2k'^2\operatorname{sn}A\operatorname{sn}B\operatorname{sn}C\operatorname{sn}D - k'^2 \\ &= 2k^2k'^2\,\frac{\operatorname{sn}^2\tfrac{1}{2}(A+B)-\operatorname{sn}^2\tfrac{1}{2}(C+D)}{1-k^2\operatorname{sn}^2\tfrac{1}{2}(A+B)\operatorname{sn}^2\tfrac{1}{2}(A-B)}\,\frac{\operatorname{sn}^2\tfrac{1}{2}(A-B)-\operatorname{sn}^2\tfrac{1}{2}(C-D)}{1-k^2\operatorname{sn}^2\tfrac{1}{2}(C+D)\operatorname{sn}^2\tfrac{1}{2}(C-D)}.\end{aligned} \tag{I}$$

If the left-hand side of (I) vanishes, we have, therefore, either

$$A+B = C+D, \qquad A+C = B+D, \qquad A+D = B+C,$$

or

$$A+B+C+D = 0.$$

If we put $C = A$, $D = B$, we get

$$\operatorname{dn}^2A\operatorname{dn}^2B - k^2\operatorname{cn}^2A\operatorname{cn}^2B + k^2k'^2\operatorname{sn}^2A\operatorname{sn}^2B = k'^2. \tag{II}$$

Taking any quadric surface $ax^2+by^2+cz^2 = 1$, it is obvious from (II) that we may write

$$x\sqrt{a} = k\,\mathrm{sn}(u+U)\mathrm{sn}(v+V), \qquad k'iy\sqrt{b} = k\,\mathrm{cn}(u+U)\mathrm{cn}(v+V),$$
$$k'z\sqrt{c} = \mathrm{dn}(u+U)\mathrm{dn}(v+V);$$

and treat u, v as the coordinates of a point on the quadric,* U, V being constants.

On writing down the equation of the tangent plane to the quadric at the point (u', v') and the condition that (u, v) should lie on it, we see from (I) that the equations of the two systems of generators on the quadric are

$$u-v = \text{constant}, \qquad u+v = \text{constant}.$$

2. I consider a pencil of hyperboloids of one sheet which have a common curve of intersection. I take all the hyperboloids to have the coordinate planes as principal planes, I suppose them to meet the plane $z = 0$ in a series of ellipses no two of which intersect, and I also suppose that no two of the asymptotic cones of the hyperboloids have any common generator.

These conditions, which have all the appearance of defining a special kind of pencil, do not, in fact, affect the generality of the pencil, when it is transformed by an arbitrary projection.

For suppose that one of the general hyperboloids has the equation $x^2+y^2 = z^2+w^2$, whilst another has one of the equations

$$\text{(i)} \quad ax^2+by^2 = cz^2+dw^2, \qquad \text{(ii)} \quad ax^2-by^2 = cz^2-dw^2,$$

a, b, c, d being taken positive. Then the curve of intersection will always meet two and only two of the planes $x = 0$, $y = 0$, $z = 0$, $w = 0$.

In case (i), the curve of intersection meets $w = 0$ in

$$x^2/(b-c) = y^2/(c-a) = z^2/(b-a),$$

so that

if $w = 0$ meet the curve, $a-b$, $a-c$, $c-b$ have the same sign;

if $z = 0$ meet the curve, $a-b$, $a-d$, $d-b$ have the same sign;

if $x = 0$ meet the curve, $d-c$, $d-b$, $b-c$ have the same sign;

if $y = 0$ meet the curve, $d-c$, $d-a$, $a-c$, have the same sign.

* To be of use, the curvilinear coordinates must be distinct and separable, like the (x, y) coordinates in plane geometry. With elliptic coordinates there are four possible categories, viz.: a, $K'i+a$, ai, $K+ai$.

Now we can arrange the order of the letters so that $a > b$, $c > d$, and $a > c$, so the possibilities are $a > b > c > d$, or $a > c > b > d$. If $a > b > c > d$, the curve of intersection does not meet any of the four planes, but in this case

$$ax^2+by^2-cz^2-dw^2-b(x^2+y^2-z^2-w^2)$$
$$= (a-b)x^2+(b-c)z^2+(b-d)w^2 > 0,$$

so the surfaces do not intersect at all. If $a > c > b > d$, the curve of intersection meets $x = 0$ and $w = 0$, but not $y = 0$ or $z = 0$.

In case (ii), by similar reasoning, after changing the sign of b and of d, we see that the curve of intersection cannot meet both $w = 0$ and $y = 0$, or both $z = 0$ and $x = 0$, but will always meet one of each pair.

I take $x = pk\,\mathrm{sn}\,u$, $k'y = qk\,\mathrm{cn}\,u$, $k'z = \pm r\,\mathrm{dn}\,u$ as the parametric equations of the curve of intersection, wherein u is real and runs from 0 to $4K$, so that $z = 0$ does not intersect the curve.

Then the hyperboloids $S(\theta)$ of the system are obtained by assigning different values to the parameter θ in the equations

$$x = pk\,\mathrm{sn}\,u\frac{\mathrm{sn}(K+iK'+v)}{\mathrm{sn}(K+iK'+\theta)}, \qquad k'y = qk\,\mathrm{cn}\,u\frac{\mathrm{cn}(K+iK'+v)}{\mathrm{cn}(K+iK'+\theta)},$$

$$k'z = r\,\mathrm{dn}\,u\frac{\mathrm{dn}(K+iK'\dot{+}v)}{\mathrm{dn}(K+iK'+\theta)};$$

wherein v is real and runs from $-K$ to K, and θ is a real parameter which also may be taken to be between $-K$ and K, as it is convenient not to restrict it further. The surfaces $S(\pm\theta)$ are the same surface; a point (u, v) on the surface $S(\theta)$ is the same as the point $(u, -v)$ on the surface $S(-\theta)$.

With this specification there is only one point with coordinates u, v on each surface $S(\theta)$; the two separate parts of the curve of intersection, considered as lying on the surface $S(\theta)$, are given by $v = \theta$, $v = -\theta$, whilst $v = 0$ gives the section of the surface $S(\theta)$ by the plane $z = 0$. The u-coordinate of a point on the curve of intersection is the same on whatever hyperboloid it is supposed to be.

Then, denoting points on the curve of intersection by their u-coordinates (taking u_1, u_3, u_5,... on one branch of the curve, and u_2, u_4, u_6,... on the other), let us consider the line joining u_r to u_{r+1}. This line will determine a hyperboloid $\theta = \theta_r$ which contains it as a

generator, and since the equations of the generators are $u \pm v = \text{constant}$, we have

$$\text{either } u_r - u_{r+1} = 2\theta_r, \quad \text{or} \quad u_r - u_{r+1} = -2\theta_r,$$

according to the system to which the generator belongs.

We distinguish $S(\theta)$ from $S(-\theta)$ by the convention that $S(\theta)$ shall represent the surface for which that part of the curve of intersection which lies above the principal plane $z = 0$ is given by $v = \theta$.

3. Consider now a skew hexagon $AC'BA'CB'$ whose vertices lie on the curve of intersection of the two hyperboloids $S(\alpha)$ and $S(-\beta)$,* which are determined by containing the three lines AC', BA', CB', and the three $C'B$, $A'C$, $B'A$, respectively. A, B, C may be taken to lie on the upper part of the curve of intersection, and A', B', C' on the lower. Denote the vertices A, C', B, A', C, B' in this order by their u-coordinates u_1, u_2, u_3, u_4, u_5, u_6. Then the equation of the line AC' in $S(\alpha)$ is

$$u - v = u_1 - \alpha = u_2 + \alpha,$$

whilst that of $A'C$ in $S(-\beta)$ is

$$u - v = u_4 + \beta = u_5 - \beta.$$

Therefore, in accordance with § 2, we shall have, multiples of $4K$ being neglected,

$$2\alpha = u_1 - u_2 = u_3 - u_4 = u_5 - u_6,$$
$$-2\beta = u_2 - u_3 = u_4 - u_5 = u_6 - u_1,$$

so that, on addition, $6(\alpha - \beta) = 4K$, or $6(\alpha - \beta) = 8K$, are the conditions that such a skew hexagon may be inscribed in the two quadrics $S(\alpha)$ and $S(-\beta)$; and we see that, if one hexagon can be so inscribed, so can any number.

This porism gives a proof of Schläfli's theorem, since a double six is made up of two such skew hexagons inscribed in the same two quadrics, the systems to which the generators belong being different in the two hexagons.

4. The Cartesian equation of the surface $S(\theta)$ is

$$\frac{x^2}{p^2}\mathrm{dn}^2\theta + k'^2\frac{y^2}{q^2} - k^2k'^2\frac{z^2}{r^2}\mathrm{sn}^2\theta = k^2\mathrm{cn}^2\theta.$$

In connexion with the surfaces $S(\alpha)$ through AC', CB', BA', and $S(-\beta)$ through $C'B$, $A'C$, $B'A$, I consider also the surface $T(\alpha, -\beta)$,

* It would have been more natural to write β for $-\beta$ here; the negative sign is introduced for the sake of uniformity in later formulae.

the Schur quadric with regard to which the opposite sides of the hexagon, for example AC' and $A'C$, are polar lines. If $\beta = \alpha \pm \frac{2}{3}K$, the equation of $T(\alpha, -\beta)$ is

$$-\frac{x^2}{p^2}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta - k'^2\frac{y^2}{q^2} - k^2k'^2\frac{z^2}{r^2}\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta = k^2\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta,$$

but, if $\beta = \alpha \pm \frac{4}{3}K$, the equation is

$$\frac{x^2}{p^2}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta + k'^2\frac{y^2}{q^2} - k^2k'^2\frac{z^2}{r^2}\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta = k^2\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta.$$

We can verify that any point (x', y', z'), or (u', v'), on AC' is conjugate to any point (x'', y'', z''), or (u'', v''), on $A'C$, with respect to $T(\alpha, -\beta)$, as follows.

The equation of AC' is $u - v = u_1 - \alpha$, and that of $A'C$ is $u - v = u_4 + \beta$, so that, since $u_1 - u_4 = 4\alpha - 2\beta$,

$$u' - v' - u'' + v'' = u_1 - u_4 - \alpha - \beta = 3(\alpha - \beta) = 2K \text{ or } 4K$$

according as $6(\alpha - \beta) = 4K$ or $8K$.

If $6(\alpha - \beta) = 4K$, $u' - v' - u'' + v'' = 2K$, we substitute in formula (I) of § (1),

$$A = u' - 2K, \quad B = u'', \quad C = K + iK' + v', \quad D = K + iK + v'',$$

and obtain

$$-\frac{x'x''}{p^2}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta - k'^2\frac{y'y''}{q^2} - k^2k'^2\frac{z'z''}{r^2}\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta = k^2\mathrm{cn}\,\alpha\,\mathrm{sn}\,\beta.$$

5. If, with the same six vertices, we now consider the hexagon $AC'CB'BA'$, for which the order of the u-coordinates is u_1, u_2, u_5, u_6, u_3, u_4, we have, multiples of $4K$ being rejected,

$$2\alpha = u_1 - u_2 = u_5 - u_6 = u_3 - u_4,$$
$$-4\beta + 2\alpha = u_2 - u_5 = u_6 - u_3 = u_4 - u_1;$$

so that taking $\beta = \alpha - \frac{2}{3}K$, $\gamma = \alpha - \frac{4}{3}K$, the three lines AC', BA', CB' lie on the hyperboloid $\theta = \alpha$, the three $C'B$, $A'C$, $B'A$ on $\theta = \beta$, and the three AA', BB', CC' on $\theta = \gamma$. Also $\beta - \gamma = \frac{2}{3}K$.

We have therefore three hexagons with the same six vertices, and three Schur quadrics. These three Schur quadrics have a common curve of intersection.* But, in fact, more than this is true.

The Schur quadrics of all skew hexagons whose vertices lie on the curve of intersection of a pencil of quadrics, and whose alternate sides are generators of two quadrics of the pencil, all have a common curve of intersection.

* Baker, *J. of London Math. Soc.* 1 (1926), 179.

For, taking $\alpha-\beta = \frac{2}{3}K$, the Schur quadric $T(\alpha, -\beta)$ is

$$-\frac{x^2}{p^2}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta - k'^2\frac{y^2}{q^2} - k^2k'^2\frac{z^2}{r^2}\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta = k^2\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta.$$

Then, from the addition-formulae

$$\mathrm{dn}\,\tfrac{2}{3}K\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta - k^2\mathrm{cn}\,\tfrac{2}{3}K\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta = k'^2$$

$$\mathrm{dn}\,\tfrac{2}{3}K - k^2\mathrm{cn}\,\tfrac{2}{3}K\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta = \mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta,$$

we see that the Schur quadric passes through the curve of intersection* of

$$k'^2\frac{z^2}{r^2} = \mathrm{cn}\,\tfrac{2}{3}K\frac{x^2}{p^2} + \mathrm{dn}\,\tfrac{2}{3}K, \qquad 1 = \mathrm{cn}\,\tfrac{2}{3}K\frac{y^2}{q^2} + \mathrm{dn}\,\tfrac{2}{3}K\frac{z^2}{r^2}.$$

6. Skew polygons of $4n+2$ sides, whose alternate sides are generators of two quadrics of a pencil

Such polygons have properties similar to those of the hexagon.

If one such polygon be given, then

(i) with the same $4n+2$ vertices, we may construct $n(2n+1)$ such polygons;

(ii) the relation between the parameters $\alpha, -\beta$ of the two quadrics of the pencil, which contain the alternate sides of the polygon, will be

$$\alpha-\beta = \pm\frac{2K}{2n+1}, \pm\frac{4K}{2n+1}, \ldots, \pm\frac{4nK}{2n+1};$$

(iii) each polygon will have a Schur quadric $T(\alpha, -\beta)$ with regard to which its opposite sides are polar lines;

(iv) these Schur quadrics will be members of n pencils, and the common curve of intersection of the pencil corresponding to

$$\alpha-\beta = \frac{2sK}{2n+1} \quad \text{or} \quad \frac{4n+2-2s}{2n+1}K$$

is given by

$$k'^2\frac{z^2}{r^2} = \mathrm{cn}\frac{2sK}{2n+1}\frac{x^2}{p^2} + \mathrm{dn}\frac{2sK}{2n+1}, \qquad 1 = \mathrm{cn}\frac{2sK}{2n+1}\frac{y^2}{q^2} + \mathrm{dn}\frac{2sK}{2n+1}\frac{z^2}{r^2}.$$

wherein s is odd.

7. The deduction, from an addition-formula for the elliptic functions, of the invariant relations between either of the quadrics

* If we change $\frac{2}{3}K$ into $\frac{4}{3}K$, we must change the signs of x^2 and y^2, and get the same result.

$S(\alpha)$, $S(\beta)$, which pass through the sets of alternate sides of the polygon, and the Schur quadric $T(\alpha, -\beta)$, and also between certain pairs of Schur quadrics is interesting. The addition-formula itself also appears to be novel. We notice, to begin with, that, according as s is odd or even in $(2n+1)(\alpha-\beta) = 2sK$, the roots of the discriminant of $S(\alpha)-\lambda T(\alpha, -\beta)$, or of $T(\alpha, -\beta)-\lambda S(\beta)$, may be taken to be either

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 1, \frac{\mathrm{dn}\,\alpha}{\mathrm{dn}\,\beta}, \frac{-\mathrm{cn}\,\alpha}{\mathrm{cn}\,\beta}, \frac{-\mathrm{sn}\,\alpha}{\mathrm{sn}\,\beta},$$

or

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 1, \frac{\mathrm{dn}\,\alpha}{\mathrm{dn}\,\beta}, \frac{\mathrm{cn}\,\alpha}{\mathrm{cn}\,\beta}, \frac{\mathrm{sn}\,\alpha}{\mathrm{sn}\,\beta},$$

and that these are also the roots of the discriminant of the two Schur quadrics, $T(\alpha, \gamma)$, $T(\beta, \gamma)$, which are determined by α, γ and β, γ; or, in particular, of two Schur quadrics, for which the corresponding polygons have one set of alternate sides in common.

8. I will first find the invariant condition for the hexagon, which is known to be $\Theta\Theta' = 4\Delta\Delta'$, or

$$(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right) = 4,$$

wherein λ_1, λ_2, λ_3, λ_4 are the roots of the discriminant of

$$S(\alpha)-\lambda T(\alpha, -\beta).$$

If $3a = 4K$, $\mathrm{sn}^2 a = \mathrm{sn}^2 \tfrac{1}{2}a = (1-\mathrm{cn}\,a)/(1+\mathrm{dn}\,a)$ or

$$(1+\mathrm{dn}\,a)(1+\mathrm{cn}\,a) = 1,$$

so that

$$1+\frac{1}{\mathrm{dn}\,\frac{1}{3}K}+\frac{1}{\mathrm{cn}\,\frac{1}{3}K} = 0, \qquad 1+\frac{1}{\mathrm{dn}\,\frac{2}{3}K}-\frac{1}{\mathrm{cn}\,\frac{2}{3}K} = 0.$$

Now consider

$$f_1(u_1) = \left(1+\frac{d_1}{d_2}-\frac{c_1}{c_2}-\frac{s_1}{s_2}\right)\left(1+\frac{d_2}{d_1}-\frac{c_2}{c_1}-\frac{s_2}{s_1}\right)$$

wherein $s_1 = \mathrm{sn}\,u_1, \ldots, s_2 = \mathrm{sn}\,u_2, \ldots$, and u_1-u_2 is constant and equal to a.

It is obvious that $f_1(u_1)$ is not altered, if u_1 and u_2 are interchanged; and, further, if we add a quarter-period, K, iK', $K+iK'$ to both u_1 and u_2 we find that $f_1(u_1)$ is not altered, the factors being reproduced, multiplied by numbers whose product is unity.

The infinities of $f_1(u_1)$ are at the quarter-periods (zeros and poles of

$\operatorname{sn} u_1$, $\operatorname{cn} u_1$, $\operatorname{dn} u_1$, $\operatorname{sn} u_2$, $\operatorname{cn} u_2$, $\operatorname{dn} u_2$), so it is sufficient to consider the residue at $u_1 = 0$, which is

$$\operatorname{sn} a\left(1+\frac{1}{\operatorname{dn} a}-\frac{1}{\operatorname{cn} a}\right).$$

Thus, if $a = \frac{2}{3}K$, the residue is zero, $f_1(u_1)$ has no pole at $u_1 = 0$, and therefore no poles at all, and, by Liouville's theorem, is a constant, which is easily found to be 4, by putting $u_1 = -u_2$.

Thus $f_1(u_1) = 4$, i.e. $\Theta\Theta' = 4\Delta\Delta'$.

9. A general addition-theorem

I take $u_1 - u_2 = a$, where, now, a has any fixed value, and write s_1 for $\operatorname{sn} u_1$, s_2 for $\operatorname{sn} u_2$, s for $\operatorname{sn} a$, etc., etc., and consider the four functions

$$f(u_1) = \left(1+\frac{d_1}{d_2}+\frac{c_1}{c_2}+\frac{s_1}{s_2}\right)\left(1+\frac{d_2}{d_1}+\frac{c_2}{c_1}+\frac{s_2}{s_1}\right),$$

$$f_1(u_1) = \left(1+\frac{d_1}{d_2}-\frac{c_1}{c_2}-\frac{s_1}{s_2}\right)\left(1+\frac{d_2}{d_1}-\frac{c_2}{c_1}-\frac{s_2}{s_1}\right),$$

$$f_2(u_1) = \left(1-\frac{d_1}{d_2}+\frac{c_1}{c_2}-\frac{s_1}{s_2}\right)\left(1-\frac{d_2}{d_1}+\frac{c_2}{c_1}-\frac{s_2}{s_1}\right),$$

$$f_3(u_1) = \left(1-\frac{d_1}{d_2}-\frac{c_1}{c_2}+\frac{s_1}{s_2}\right)\left(1-\frac{d_2}{d_1}-\frac{c_2}{c_1}+\frac{s_2}{s_1}\right).$$

Each function is unaltered if we interchange u_1 and u_2, or if we add a quarter-period to both u_1 and u_2, so that, after determining the residues at $u_1 = 0$, we find that Liouville's theorem gives us

$$\frac{f(u_1)-4}{-\left(1+\frac{1}{d}+\frac{1}{c}\right)} = \frac{f_1(u_1)-4}{1+\frac{1}{d}-\frac{1}{c}} = \frac{f_2(u_1)-4}{1-\frac{1}{d}+\frac{1}{c}} = \frac{f_3(u_1)-4}{-\left(1-\frac{1}{d}-\frac{1}{c}\right)}.$$

10. The Decagon

If $5a = 4K$, or $3a = 4K$, then $\operatorname{sn}^2 2a = \operatorname{sn}^2 \frac{1}{2}a$,

i.e. $$\frac{4s^2c^2d^2}{(1-k^2s^4)^2} = \frac{1-c}{1+d},$$

i.e. $$4(1+d)(1+c)c^2d^2 = (c^2+d^2-c^2d^2)^2,$$

i.e. $$\frac{4}{cd}\left(1+\frac{1}{c}+\frac{1}{d}\right) = \frac{1}{c^4}+\frac{1}{d^4}+1-\frac{2}{c^2}-\frac{2}{d^2}-\frac{2}{c^2d^2},$$

so that dividing by $1+\frac{1}{c}+\frac{1}{d}$ we have, if $5a = 4K, 8K, 12K, 16K,$

$$\frac{4}{cd} = \left(1+\frac{1}{d}-\frac{1}{c}\right)\left(1-\frac{1}{d}+\frac{1}{c}\right)\left(1-\frac{1}{d}-\frac{1}{c}\right)$$

and, if $5a = 2K, 6K, 14K, 18K$

$$-\frac{4}{cd} = \left(1+\frac{1}{d}+\frac{1}{c}\right)\left(1-\frac{1}{d}-\frac{1}{c}\right)\left(1-\frac{1}{d}+\frac{1}{c}\right),$$

which may be written

$$\frac{1}{1+\frac{1}{d}+\frac{1}{c}} + \frac{1}{1-\frac{1}{d}-\frac{1}{c}} + \frac{1}{-1+\frac{1}{d}-\frac{1}{c}} + \frac{1}{1+\frac{1}{d}-\frac{1}{c}} = 0.$$

It follows from this, that if $u_1-u_2 = \frac{2}{5}K$,

$$\frac{1}{f(u_1)-4}+\frac{1}{f_2(u_1)-4}+\frac{1}{f_3(u_1)-4} = \frac{1}{f_1(u_1)-4},$$

so that, if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of the discriminant, and we write

$$\phi(x,y,z,w) \equiv (x+y+z+w)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)-4,$$

the invariant relation for, say, the Schur quadrics of two skew decagons, each of which has one set of alternate sides inscribed in the same quadric, whilst the other sets are inscribed in two other quadrics of the pencil, is

$$\frac{1}{\phi(\lambda_1,\lambda_2,-\lambda_3,-\lambda_4)}+\frac{1}{\phi(\lambda_1,-\lambda_2,\lambda_3,-\lambda_4)}+\frac{1}{\phi(\lambda_1,-\lambda_2,-\lambda_3,\lambda_4)} = \frac{1}{\phi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}.$$

Of these four ϕ-functions, $\phi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is $(\Theta\Theta'-4\Delta\Delta')/\Delta\Delta'$ in the ordinary notation, and the other three are roots of a cubic,* with

* If the discriminant equation is written, with binomial coefficients, in the form

$$a\lambda^4+4b\lambda^3+6c\lambda^2+4d\lambda+e = 0,$$

I find that the three functions ϕ, multiplied by ae, i.e. $\Delta\Delta'$, are the roots of

$$(x+4I+6c^2)(x-4I+12c^2)^2 = 2(3cx+4cI+18c^3+32J)^2;$$

each of them being a quadratic function of a root of the reducing cubic. In fact, if the quartic equation is written

$$(\lambda^2+2p\lambda+q)(\lambda^2+2p'\lambda+q') = 0,$$

then $$\theta \equiv p^2q'+p'^2q = (p+p')(pq'+p'q)-pp'(q+q')$$

has only three values, and both pp' and $q+q'$ are linear functions of a root of the reducing cubic $4a^3x^3-Iax+J = 0$.

coefficients rational in $\Theta, \Theta', \Phi, \Delta, \Delta'$, the first coefficient being given by

$$\phi(\lambda_1, \lambda_2, -\lambda_3, -\lambda_4)+\phi(\lambda_1, -\lambda_2, \lambda_3, -\lambda_4)+\phi(\lambda_1, -\lambda_2, -\lambda_3, \lambda_4) = -\phi(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

11. The invariant relation associated with a skew polygon of $4n+2$ sides, whose alternate sides are generators of two quadrics, can be investigated in a similar manner. We must first show that there is a polynomial in $\operatorname{nc} a$, $\operatorname{nd} a$, of degree $\frac{1}{2}n(n+1)$, which vanishes when $(2n+1)a = 4sK$. Now $\operatorname{sn}^2 na = \operatorname{sn}^2 \frac{1}{2}a$, if $(2n+1)a = 4sK$, or $(2n-1)a = 4sK$, so that it is sufficient to deduce from $\operatorname{sn}^2 na = \operatorname{sn}^2 \frac{1}{2}a$, an equation in $\operatorname{nc} a$, $\operatorname{nd} a$, of degree n^2, since $\frac{1}{2}n(n+1)+\frac{1}{2}(n-1)n = n^2$. Consider the two cases, n even, and n odd, separately.

We first prove that

(i) when n is even,

$$\operatorname{sn} nu = \frac{2s}{cd}\frac{S_n}{N_n}, \qquad \operatorname{cn} nu = \frac{C_n}{N_n}, \qquad \operatorname{dn} nu = \frac{D_n}{N_n};$$

wherein S_n, C_n, D_n, N_n are polynomials in c^{-1}, d^{-1}, even in both variables, in which the coefficients are integers, and so independent of k. The degree of C_n, D_n, N_n is $\frac{1}{2}n^2$ and of S_n is $\frac{1}{2}n^2-2$;

(ii) when n is odd,

$$\operatorname{sn} nu = \frac{sS_n}{N_n}, \qquad \operatorname{cn} nu = \frac{cC_n}{N_n}, \qquad \operatorname{dn} nu = \frac{dD_n}{N_n},$$

wherein S_n, C_n, D_n, N_n are similar polynomials in c^{-1}, d^{-1}, each of degree $\frac{1}{2}(n^2-1)$.

Since

$$\operatorname{sn} 2u = \frac{2scd}{c^2+d^2-c^2d^2}, \qquad \operatorname{cn} 2u = \frac{c^2-d^2+c^2d^2}{c^2+d^2-c^2d^2},$$

$$\operatorname{dn} 2u = \frac{-c^2+d^2+c^2d^2}{c^2+d^2-c^2d^2},$$

this is true when $n = 1, 2$.

We can now prove the general result by induction, for, if we assume the result true for $n = m$, $m+1$, it is also true for $n = 2m$, by the use of the above formulae for $\operatorname{sn} 2u$, etc.; and also for $n = 2m+1$, by the use of the addition-formula, which gives, for example,

$$\begin{aligned} N_{2m+1} &= N_m^2 N_{m+1}^2 - 4k^2(s^4/c^2d^2)S_m^2 S_{m+1}^2 \\ &= N_m^2 N_{m+1}^2 - 4(1-c^{-2}-d^{-2}+c^{-2}d^{-2})S_m^2 S_{m+1}^2. \end{aligned}$$

Thus, as it is true for $n = 1, 2$, it is true generally. We notice further, than when n is odd,

$$N_n^2 = c^2C_n^2+s^2S_n^2, \qquad N_n^2 = d^2D_n^2+ks^2S_n^2,$$

which gives $N_n^2-S_n^2 = c^2(C_n^2-S_n^2)$, $N_n^2-S_n^2 = d^2(D_n^2-S_n^2)$, so that $C_n^2-S_n^2$, $D_n^2-S_n^2$, have c^{-2}, d^{-2}, as factors, respectively, and the degree of $N_n^2-S_n^2$ in c^{-1}, d^{-1} is therefore reduced by two, and is n^2-3.

Thus, (i) when n is even, the equation $\mathrm{sn}^2na = \mathrm{sn}^2\frac{1}{2}a$, becomes

$$4s^2S^2/c^2d^2 = N^2(1-c)/(1+d),$$

i.e.
$$\frac{4}{cd}\left(1+\frac{1}{c}\right)\left(1+\frac{1}{d}\right)S^2 = N^2,$$

which is of the required form, since S is of degree $\frac{1}{2}n^2-2$, and N of degree $\frac{1}{2}n^2$ in c^{-1}, d^{-1};

(ii) when n is odd, $\mathrm{sn}^2na = \mathrm{sn}^2\frac{1}{2}a$, gives

$$(1+c)(1+d)S^2 = N^2,$$

i.e.
$$\left(1+\frac{1}{c}+\frac{1}{d}\right)S^2 = \frac{1}{cd}(N^2-S^2),$$

which again is of the required form, since S is now of degree $\frac{1}{2}(n^2-1)$, and N^2-S^2 is of degree n^2-3.

By using the appropriate transformations viz:

$$\mathrm{cn}(u,k) = \mathrm{nc}(iu,k'), \qquad \mathrm{dn}(u,k) = \mathrm{dc}(iu,k');$$
$$\mathrm{cn}(u,k) = \mathrm{dn}(ku,k^{-1}), \quad \mathrm{dn}(u,k) = \mathrm{cn}(ku,k^{-1});$$

it is easy to see that the polynomials in $\mathrm{nc}\,a$, $\mathrm{nd}\,a$ obtained in this way for $(2n+1)a = 4K$ are symmetrical in the three variables 1, $\mathrm{nc}\,a$, $\mathrm{nd}\,a$.

It follows that the invariant condition associated with the polygon of $4n+2$ sides, is the vanishing of a symmetrical function of $\phi(\lambda_1,\lambda_2,-\lambda_3,-\lambda_4)$, $\phi(\lambda_1,-\lambda_2,\lambda_3,-\lambda_4)$, $\phi(\lambda_1,-\lambda_2,-\lambda_3,\lambda_4)$, and so, finally,* of a function of c, I, J, or, in the more usual notation, of

$$\Phi, \qquad \begin{vmatrix} 2\Delta, & \Theta \\ \Theta', & 2\Delta' \end{vmatrix}, \qquad \begin{vmatrix} 12\Delta, & 3\Theta, & 2\Phi \\ 3\Theta, & 2\Phi, & 3\Theta' \\ 2\Phi, & 3\Theta', & 12\Delta' \end{vmatrix}.$$

I have to thank Mr. J. Hodgkinson for reading the first draft of this paper in manuscript. His criticisms and suggestions have led to many improvements.

* In the notation of § 10, p. 68, footnote.

Postscript

Something should have been said about the intimate connexion of the theorems of this paper with the corresponding theorems in plane geometry.

In the first place we have an obvious analogue of Poncelet's general theorem, which is: If all the sides but one of any polygon inscribed in a fixed conic of a pencil touch fixed conics of the pencil, then the remaining side will also touch a fixed conic of the pencil. The analogue is: If all the sides but one of any skew polygon whose vertices lie on a quadri-quadric curve lie in fixed quadrics of the pencil through the curve, then the remaining side will also lie in a fixed quadric of the pencil.

But the problem solved in this paper is not only analogous to the corresponding plane problem of the inscribed and circumscribed polygon, but is, in some sense, identical with it.

I have found above the invariant relation corresponding to $\alpha-\beta = 2sK/n$, by the use of an auxiliary relation for the special case when $\beta = 0$. But, geometrically, if $\beta = 0$, the hyperboloid $S(\beta)$ has the point equation $x^2/p^2+k'^2y^2/q^2 = k^2$, whilst its tangential equation shows that $S(0)$ is a cylinder with generators parallel to the axis of z. Also the condition for the existence of a skew polygon of $4n+2$ sides, with one set of alternate sides in $S(0)$, that is to say parallel to the axis of z, and the other set in $S(\alpha)$, is just the condition for the existence of a plane polygon of $2n+1$ sides, which is inscribed in

$$\frac{x^2}{p^2}+k'^2\frac{y^2}{q^2} = k^2,$$

and circumscribed to

$$\frac{x^2}{p^2}\mathrm{dn}^2\alpha+k'^2\frac{y^2}{q^2} = k^2\mathrm{cn}^2\alpha.$$

It follows that the auxiliary relations which I have used are, in fact, the invariant relations for such a system of Poncelet's poristic polygons.

Conversely, from the properties of a porism of skew polygons whose vertices lie on a quadri-quadric curve, we can derive those of a porism of plane polygons. To do this, take the common self-conjugate tetrahedron of the pencil of quadrics, and consider the trace of the three-dimensional figure on a face of this tetrahedron. The plane through two consecutive edges of the skew polygon is a tangent plane to two fixed quadrics, and will therefore meet the face of the

tetrahedron in a tangent to a fixed conic; whilst the sides of the skew polygon will meet the face in points on fixed conics.

Take, for example, the skew polygon of $4n+2$ sides, which are, alternately, generators of $S(\alpha)$ and $S(\beta)$, whose opposite sides, therefore, are polar lines with regard to $T(\alpha_1-\beta)$. Take

$$(2n+1)(\alpha-\beta) = 4sK.$$

Then the trace on the plane $z = 0$ of this three-dimensional figure is a plane polygon whose vertices lie, alternately, on

$$\frac{x^2}{p^2}\mathrm{dn}^2\alpha+k'^2\frac{y^2}{q^2} = k^2\mathrm{cn}^2\alpha, \qquad \frac{x^2}{p^2}\mathrm{dn}^2\beta+k'^2\frac{y^2}{q^2} = k^2\mathrm{cn}^2\beta;$$

and whose sides touch

$$\frac{x^2}{p^2}\mathrm{dn}^2\alpha\,\mathrm{dn}^2\beta+k'^2\frac{y^2}{q^2} = k^2\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta.$$

It will follow that the opposite vertices of this polygon are conjugate points with regard to

$$\frac{x^2}{p^2}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta+k'^2\frac{y^2}{q^2} = k^2\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta,$$

or, in other words, that the polar, with regard to this conic, of any vertex on one conic is the tangent to the other conic at the opposite vertex. This seems to be a new property.

I find that T. W. Chaundy has, in two papers,* made a systematic study of Poncelet's poristic polygons with 3, 5,... or almost any number of sides, using his own notation for elliptic functions, which is not quite the same as Jacobi's; I might have quoted the formulae for the triangle and pentagon from his papers, but as the notation is different, and, also, to some extent, the method of treatment, I have preferred to leave my own work as it stands.

A NOTE ON A FORMULA FOR $\mathrm{sn}(2n+1)u$

From the special formulae in elliptic functions used in the above paper, an interesting deduction may be made.

Since, $3u = 4K$, it follows (§ 8) that

$$\mathrm{cn}\,u\,\mathrm{dn}u+\mathrm{cn}\,u+\mathrm{dn}\,u = 0,$$

it is readily inferred that the numerator of $\mathrm{sn}\,3u$, when it is expressed

* *Proc. London Math. Soc.* (2) 22 (1923), 104–23; 25 (1926), 17–44.

in elliptic functions of u, is equal to

$$\operatorname{sn} u \prod (\operatorname{cn} u \operatorname{dn} u \pm \operatorname{cn} u \pm \operatorname{dn} u),$$

and this can easily be verified from the known value*

$$3 \operatorname{sn} u - 4(1+k^2)\operatorname{sn}^3 u + 6k^2 \operatorname{sn}^5 u - k^4 \operatorname{sn}^9 u.$$

Similarly, if $\operatorname{sn}(2n+1)u / \operatorname{sn} u$ is expressed as an algebraic function of $\operatorname{nc} u$, $\operatorname{nd} u$, the numerator is a product of four polynomials of degree $\frac{1}{2}n(n+1)$, each of them got from any other by changing the signs of $\operatorname{cn} u$ or of $\operatorname{dn} u$, or of both. For $\operatorname{sn} 5u / \operatorname{sn} u$, two of these polynomials are given in § 10 of the above paper.

* Whittaker and Watson, *Modern Analysis*, 530, Ex. 16.

THE CONVERGENCE OF SEQUENCES DEFINED BY QUADRATIC RECURRENCE-FORMULAE

By T. W. CHAUNDY and ERIC PHILLIPS (*Oxford*)

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SEQUENCES generated by recurrence-formulae

$$u_{n+1} = \phi(u_n, u_{n-1}, \ldots),$$

which form the subject-matter of the calculus of finite differences are also important in genetic theory: an example of the latter has been discussed in this Journal by V. A. Bailey.* But, whereas in the calculus we wish to solve the recurrence-equation by expressing u_n as a function of n and sufficient initial terms u_0, u_1,..., in genetic theory we are often content to know merely the 'ultimate' behaviour of the sequence, i.e. whether it converges or how it oscillates.

Where, as, for example, in the linear recurrence-formula with constant coefficients, the finite-difference problem is solved, the ultimate behaviour of u_n presents little difficulty. Elsewhere little seems to be known about the convergence problem except in certain examples of the two-term recurrence-formula

$$u_{n+1} = \phi(u_n)$$

in which it is easy to see that there is monotonic convergence: the sequence given by Newton's method of approximation is of this type.†

There seems, then, to be a case for a more systematic study of the convergence of sequences generated by recurrence-formulae of specified types, and as an elementary contribution to this study we have considered the sequence defined by the two-term quadratic recurrence-formula

$$u_{n+1} = au_n^2 + 2bu_n + c, \tag{1}$$

where a, b, c are real and independent of n. The substitution

$$u'_n = au_n + b, \qquad c' = ac - b^2 + b \tag{2}$$

reduces (1) to the simpler form

$$u'_{n+1} = u'^2_n + c',$$

which, dropping accents, we write as

$$u_{n+1} - u_n = (u_n - k)(u_n - 1 + k). \tag{3}$$

* Bailey, *Quart. J. of Math.* (Oxford), 2 (1931), 68–77.

† Cf. for instance, Chaundy, *Differential Calculus* (Oxford, 1935), 190–3.

Now, if u_n converges, to u say, then, by taking limits in (3), we see that

$$\textit{the limit } u \textit{ can only be } k \textit{ or } 1-k. \tag{4}$$

In particular, if any u_n is k or $1-k$, every subsequent u_n has the same value and the sequence is *at length stationary*: we shall regard such a sequence as being convergent. Thus, if $|u_0| = k$, the sequence is *at length* stationary with $u_n = k$ $(n \geqslant 1)$. For all other values of u_0 the behaviour of the sequence can be summed up in the propositions:

(i) if k is not real, $u_n \to \infty$ monotonically;

(ii) if $|u_0| > k$, $u_n \to \infty$ monotonically;

(iii) if $|u_0| < k$ and $\frac{1}{2} \leqslant k \leqslant \frac{3}{2}$, $u_n \to 1-k$: monotonically if $\frac{1}{2} \leqslant k \leqslant 1$;

(iv) if $|u_0| < k$ and $\frac{3}{2} < k \leqslant 2$, u_n oscillates finitely;

(v) if $|u_0| < k$ and $k > 2$, $u_n \to \infty$, except when u_0 belongs to a certain set S, and then u_n oscillates finitely. This set has the power of the continuum and measure zero.

Since k, $1-k$ are indistinguishable in (3), we may sufficiently take $k \geqslant \frac{1}{2}$ for real k, and the above five propositions are therefore complete. We proceed to their proof.

Proof of (i). Since u_n is real, it can converge only to a real limit, and so, by (4), there cannot be convergence unless k is real. Again, if k, $1-k$ are conjugate imaginaries, the right-hand side of (3) is always positive. Thus u_n ascends monotonically: to infinity, since it does not converge.

Proof of (ii). We can rewrite (3) as

$$u_{n+1}-k = u_n^2-k^2,$$

and so
$$u_1 > k \quad \text{if} \quad |u_0| > k.$$

Again by (3), if, for some $n = r$,

$$u_n > k \ (\geqslant 1-k, \text{ since } k \geqslant \tfrac{1}{2}),$$

then
$$u_{n+1} > u_n > k,$$

and so inductively for every n greater than r. Thus u_n is monotonic ascending, and it cannot converge, since it exceeds both the possible limits k, $1-k$. It, therefore, diverges monotonically to infinity.

Proof of (iii). Consider first the case $\frac{1}{2} \leqslant k \leqslant 1$. Now we may rewrite (3) in either of the forms

$u_{n+1}-k=(u_n-k)(u_n+k)$, $u_{n+1}-1+k=(u_n-1+k)(u_n+1-k)$. (5)

The first of these shows that $|u_n|<k$ gives $0\leqslant u_{n+1}<k$, and hence, by induction, $|u_0|<k$ gives $0\leqslant u_n<k$ for every positive n.

Thus, since $k\leqslant 1$,

$$u_n+1-k, \qquad k-u_n$$

are always positive,* and so, from the second form of (5), $u_{n+1}-1+k$, u_n-1+k have always the same sign. Hence, by induction, u_n-1+k has a constant sign for every positive n, and so, by (3) itself, $u_{n+1}-u_n$ has a constant sign opposite to that of u_n-1+k. Hence u_n moves monotonically towards $1-k$ and so converges monotonically to $1-k$.

Suppose on the other hand that $1<k\leqslant\frac{3}{2}$. Now

$$\begin{aligned} u_{n+2}-u_n &= (u_{n+2}-u_{n+1})+(u_{n+1}-u_n) \\ &= (u_{n+1}-k)(u_{n+1}+k-1)+(u_n-k)(u_n+k-1), \quad \text{by (3)} \\ &= (u_n-k)(u_n+k-1)\{(u_n+k)(u_n-k+1)+1\}, \quad \text{by (5)}. \end{aligned}$$

But $$\begin{aligned}(u_n+k)(u_n-k+1)+1 &= (u_n+\tfrac{1}{2})^2-(k+\tfrac{1}{2})(k-\tfrac{3}{2}) \\ &> 0, \quad \text{since } 1<k\leqslant\tfrac{3}{2};\end{aligned}$$

unless, exceptionally, $k=\frac{3}{2}$, $u_n=-\frac{1}{2}$, i.e. $u_n=1-k$ and the sequence is at length stationary. Hence, otherwise

$$u_{n+2}-u_n \textit{ has the sign of } (u_n-k)(u_n+k-1). \tag{6}$$

Suppose now that, for some r,

$$-k<1-k<u_n<k-1<k,$$

then, by (5), $$u_{n+1}<1-k<k-1<k,$$

so that, again by (5), $1-k<u_{n+2}$, while, by (6), $u_{n+2}<u_n$. Thus

$$-k<1-k<u_{n+2}<k-1<k.$$

Then, by (5), $u_{n+3}<1-k$, while, by (6), $u_{n+3}>u_{n+1}$,

i.e. $$u_{n+1}<u_{n+3}<1-k.$$

Suppose now that $|u_0|<k-1$. Then, by induction, we see that the subsequences $\{u_{2n}\}$, $\{u_{2n+1}\}$ are both monotonic, and descend and ascend respectively, being separated by $1-k$, to which, as the only available limit, they must both converge.

It remains to consider the case $k-1<|u_0|<k$, where still $1<k\leqslant\frac{3}{2}$. By (5), $1-k<u_1<k$, i.e. $1-k<u_1<k-1$ (which is the case just considered), or else $k-1\leqslant u_1<k$. Now $u_1=k-1$ gives the stationary case $u_n=1-k$, while, if

$$1-k<k-1<u_n<k,$$

* If we exclude the stationary case $k=1$, $u_n=0$.

then $$1-k < u_{n+1} < u_n < k, \quad \text{by (3), (5).}$$

Thus, if $k-1 < |u_0| < k$, u_n diminishes from u_1 onwards, at least until it is no longer greater than $k-1$. This it must be eventually, for otherwise $\{u_n\}$ would be a monotonic descending sequence lying in the interval $(k-1, k)$ and therefore converging to some limit u such that $1-k < k-1 \leqslant u < k$, which we have already seen to be impossible. Thus eventually $u_n = k-1$ and the sequence becomes stationary; or else $1-k < u_n < k-1$, and the sequence then converges in the manner described above. This completes the proof of (iii).

Proof of (iv). Here $\frac{3}{2} < k \leqslant 2$, $|u_0| < k$. Now, if $k = 2$, the recurrence-formula is satisfied by $u_n = 2\cos 2^n\theta$ where $\theta = \cos^{-1}(\frac{1}{2}u_0)$. The sequence oscillates, except in the cases $u_0 = 2\cos(r\pi/2^s)$ where r, s are positive integers, when it is at length stationary.

If $\frac{3}{2} < k < 2$, we have from (3)

$$u_{n+1}+k = u_n^2-k(k-2) > 0, \qquad u_{n+1}-k = u_n^2-k^2.$$

Hence $|u_{n+1}| < k$, if $|u_n| < k$; and so, by induction, $|u_n| < k$ for all n, if $|u_0| < k$. The sequence is therefore bounded. Now suppose it convergent, the only possible limits being, of course, k, $1-k$. Then, by (5),

$$\left|\frac{u_{n+1}+k-1}{u_n+k-1}\right| = |u_n-k+1| \to 2k-2 > 1,$$

or else

$$\left|\frac{u_{n+1}-k}{u_n-k}\right| = |u_n+k| \to 2k > 3,$$

which makes $|u_n+k-1|$ or $|u_n-k|$ divergent. This contradiction shows that u_n cannot converge; and we have seen that it is bounded. It therefore oscillates finitely.

Proof of (v). We may reverse the recurrence-formula (3) and write

$$u_n = \pm\sqrt{(u_{n+1}+\kappa)} \qquad (\kappa \equiv k^2-k). \tag{7}$$

Thus, at length,

$$u_0 = \pm\sqrt{\{\kappa\pm\sqrt{[\kappa\pm\ldots\pm\sqrt{(\kappa\pm u_n)}\ldots]}\}}. \tag{8}$$

Using ϵ_r for a definite one or other of ± 1, we consider functions of the type

$$P\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n; x\} \equiv \sqrt{\{\kappa+\epsilon_1\sqrt{[\kappa+\epsilon_2\sqrt{\ldots+\epsilon_n\sqrt{(\kappa+x)}\ldots]}\}}}, \tag{9}$$

writing $P_n(x)$ for brevity when the set $(\epsilon_1, \ldots, \epsilon_n)$ is presumed known.* We complete the definition with $P_0(x) \equiv \sqrt{(\kappa+x)}$.

* Here and throughout $\sqrt{}$ means the positive value of the root.

Now, if $|u_n| > k$ for some n, then, by (ii), the sequence after this value diverges monotonically to infinity: this is covered by the first part of (v). We have thus only to consider the case (if any) in which $|u_n| < k$ for every n.

If $|x| \leqslant k$, then $0 < \kappa+x \leqslant k^2$, since $k > 2$, so that $\sqrt{(\kappa+x)}$ is real and, in its turn, not greater than k. Thus, at length, each $P_n(x)$ is real (and, by definition, positive) in the closed interval $(-k, k)$ and, in that interval, is evidently a monotonic function of x, so that $P_n(x)$ lies between $P_n(k)$ and $P_n(-k)$. We should also observe that the greatest $P(x)$ is

$$k = P_0(k) = P\{+1; k\} = \ldots = P\{+1, +1, \ldots, +1; k\},$$

and the least

$$\sqrt{(\kappa-k)} = P_0(-k) = P\{-1; k\} = \ldots = P\{-1, +1, \ldots, +1; k\}.$$

We now prove the lemma

$$|P_n(k)-P_n(-k)|, \textit{ for large } n, \textit{ has the order } o(2^{-n}).$$

Write $$p_n \equiv P_n(k), \qquad q_n \equiv P_n(-k),$$

and more generally, for given P_n,

$$p_r, q_r \equiv P\{\epsilon_{n-r+1}, \ldots, \epsilon_n; \pm k\} \quad (1 \leqslant r \leqslant n),$$

$$p_0, q_0 \equiv \sqrt{(\kappa \pm k)}.$$

From the definition of P_n we have

$$p_{r+1}^2 = \kappa+\epsilon_{n-r}p_r, \qquad q_{r+1}^2 = \kappa+\epsilon_{n-r}q_r,$$

so that $$|p_{r+1}-q_{r+1}| = \frac{|p_r-q_r|}{p_{r+1}+q_{r+1}},$$

while $p_0^2-q_0^2 = 2k$. Thus

$$|p_n-q_n| = (p_0-q_0) \Big/ \prod_{r=1}^{n} (p_r+q_r),$$

and so, by Hölder's inequality,

$$|p_n-q_n| < 2 \Big/ \left\{ \left(\prod_{r=1}^{n} p_r \right)^{1/n} + \left(\prod_{r=1}^{n} q_r \right)^{1/n} \right\}^n, \tag{10}$$

since $$p_0-q_0 = \frac{p_0^2-q_0^2}{p_0+q_0} = \frac{2k}{p_0+q_0} < 2.$$

Again $$(q_{r+1}^2-\kappa)^2 = q_r^2$$

gives on rearrangement

$$q_{r+1}^2 = K^2 + \frac{k^2-q_r^2}{k^2-q_{r+1}^2},$$

where $K \equiv k^2-2k > 0$, since $k > 2$. Thus

$$\prod_{r=1}^{n} q_r^2 = \prod_{r=1}^{n} \left(K^2+\frac{k^2-q_{r-1}^2}{k^2-q_r^2}\right),$$

whence, by a second application of Hölder's inequality,

$$\left(\prod_{r=1}^{n} q_r^2\right)^{1/n} > K^2+\left(\frac{k^2-q_0^2}{k^2-q_n^2}\right)^{1/n} \geqslant K^2+1, \tag{11}$$

since $q_0^2 \leqslant q_n^2 < k^2$.

A similar argument applies to the p's, except that here we may have, for some r ($\leqslant n$),

$$p_s = k \quad (1 \leqslant s \leqslant r).$$

Hölder's inequality gives in such a case

$$\left(\prod_{r=1}^{n} p_r\right)^{1/n} > k^{r/n}\,(K^2+1)^{(n-r)/2n} > (K^2+1)^{\frac{1}{2}}, \tag{12}$$

since $k^2 > K^2+1$. Then (11), (12) give in (10)

$$|p_n-q_n| < 2/2^n(K^2+1)^{n/2} = o(2^{-n}), \tag{13}$$

and the lemma is proved.*

If now we have an infinite sequence $\{\epsilon_n\}$, we can define by means of it the infinite sequence $\{P_n(x)\}$, where $P_n(x) \equiv P\{\epsilon_1,\ldots,\epsilon_n; x\}$. Then

$$P_{n+r}(x) = P_n(X),$$

where $$X \equiv \epsilon_{n+1} P\{\epsilon_{n+2},\ldots,\epsilon_{n+r}; x\}.$$

But every P_r is positive and does not exceed k, and so X lies in $(-k, k)$ and therefore every $P_{n+r}(x)$ of the sequence lies between $P_n(k)$ and $P_n(-k)$. Thus, by the lemma,

$$|P_{n+r}(x)-P_n(x)| < |p_n-q_n| = o(2^{-n}).$$

Hence $P_n(x)$ converges for every x in $(-k, k)$: and to the same limit for every x, since again

$$|P_n(y)-P_n(x)| < |p_n-q_n| = o(2^{-n}).$$

Thus each sequence $\{\epsilon_n\}$ defines in this way a precise limit, which we may denote by

$$P_\infty \equiv \sqrt{\{\kappa+\epsilon_1\sqrt{[\kappa+\epsilon_2\sqrt{(\kappa+\epsilon_3\sqrt{}\ldots \text{ to } \infty\ldots)}]}\}}. \tag{14}$$

Now, for a sequence in which every $|u_n| < k$, we can extend the formula (8) without limit. This defines an infinite sequence $\{\epsilon_n\}$ and at the same time always expresses $|u_0|$ as a $P_n(u_{n+1})$ for increasing n. Thus $|u_0|$ always lies in the closed interval (p_n, q_n) and so is itself

* Actually we have the more stringent inequality $|p_n-q_n| = o(k^{-n})$, but (13) is enough for our purpose.

the value of the limit P_∞ defined by $\{\epsilon_n\}$; and accordingly the exceptional cases in (v) are exactly those in which u_0 is one of the set

$$\pm\sqrt{\{\kappa\pm\sqrt{[\kappa\pm\sqrt{(\kappa\pm\sqrt{\ldots}\text{ to }\infty\ldots)}]}\}}. \tag{15}$$

This is the set we have called S. It has the power of the continuum, for we may correlate its terms with numbers in the interval $(0, 1)$ by writing these latter as infinite radix-factions in scale 2, the digits 1, 0 corresponding respectively (say) to $+$, $-$ in (15).

To determine the measure of S, consider the set of all the closed intervals (p_n, q_n). These intervals contain all the points $P_n(x)$ and *a fortiori* all the points $P_{n+r}(x)$, and therefore also all the limit-points P_∞, i.e. all points of the set S. Thus the measure of S does not exceed the aggregate length of these intervals. But they are 2^n in number, and, by the lemma, the length of any one is $o(2^{-n})$. By proceeding to the limit $n \to \infty$, we see that the measure of S is zero.

We should notice the case in which, after a certain stage in (15), the signs (i) are all the same; or more generally (ii) recur (in a period of r terms). The corresponding radix-fractions in scale 2 are, of course, (i) the rational fractions with denominator a power of 2; (ii) the rational fractions (with denominator a factor of 2^r-1). The corresponding sequences $\{u_n\}$ are (i) at length *stationary* at the respective values k (signs all positive) or $1-k$ (signs all negative); (ii) at length *recurring* (with period r), i.e. after some stage $u_{n+r} = u_n$ for every n.

In all other cases, i.e. when the radix-fraction represents an irrational number, the sequence of signs in (15) is also 'irrational', and the sequence $\{u_n\}$ oscillates also 'irrationally'. Thus, wherever $|u_0|$ belongs to the set S, the sequence $\{u_n\}$ oscillates (or is at length stationary), and the oscillation is bounded since, by hypothesis, $|u_n| < k$.